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MODULE 7

MATHEMATICS 31

The Integral

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Mathematics 31

Module 7

THE INTEGRAL

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Mathematics 31
Student Module Booklet
Module 7
The Integral
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Welcome

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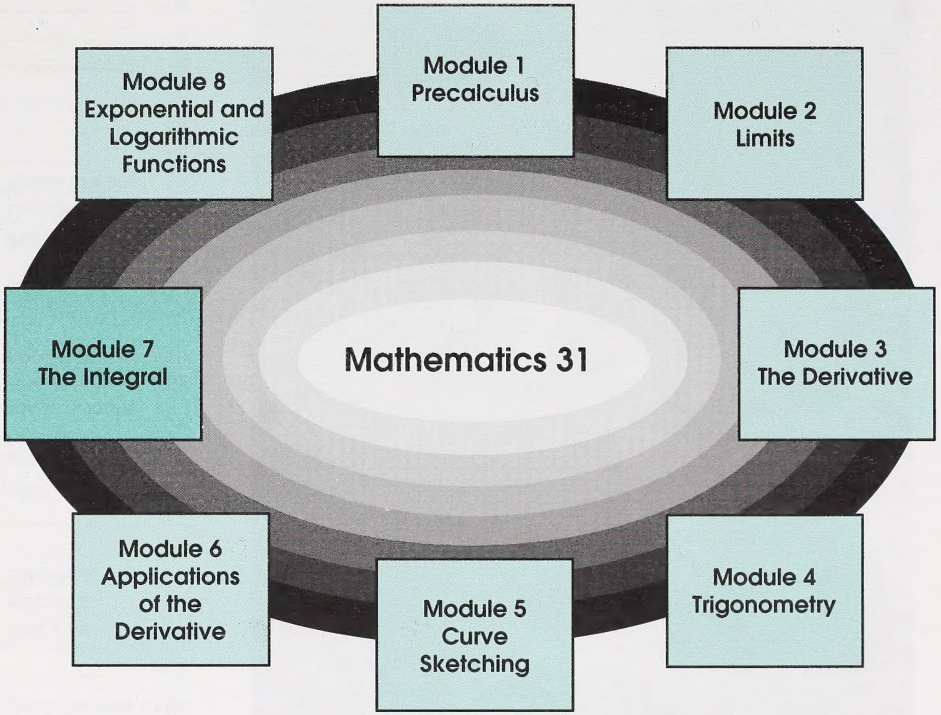
Welcome to Module 7. We hope you'll enjoy your study of The Integral.

Mathematics 31 contains eight modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.



The document you are presently reading is called a Student Module Booklet. You may find visual cues or icons throughout it. Read the following explanations to discover what each icon prompts you to do.

- Use your graphing calculator.

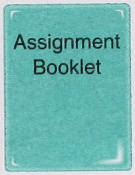
- Use your scientific calculator.

- Use computer software.

- Use the suggested answers in the Appendix to correct the activities.

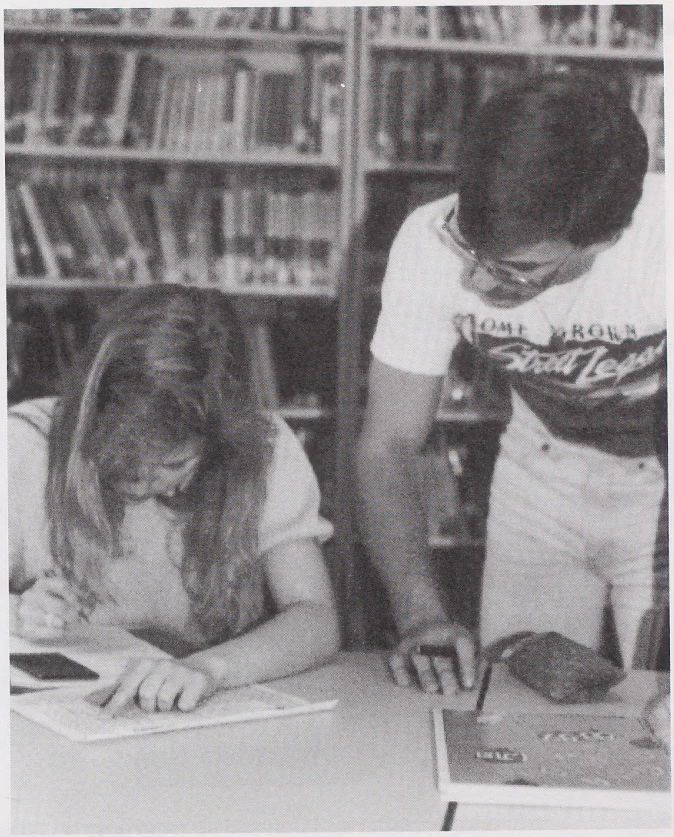
- View a videocassette.

- Pay close attention to important words or ideas.



- Answer the questions in the Assignment Booklet.

Note: Whenever the scientific calculator icon appears, you may use a graphing calculator instead.



There are no response spaces provided in this Student Module Booklet. This means that you will need to use your own paper for your responses. You should keep your response pages in a binder so that you can refer to them when you are reviewing or studying.

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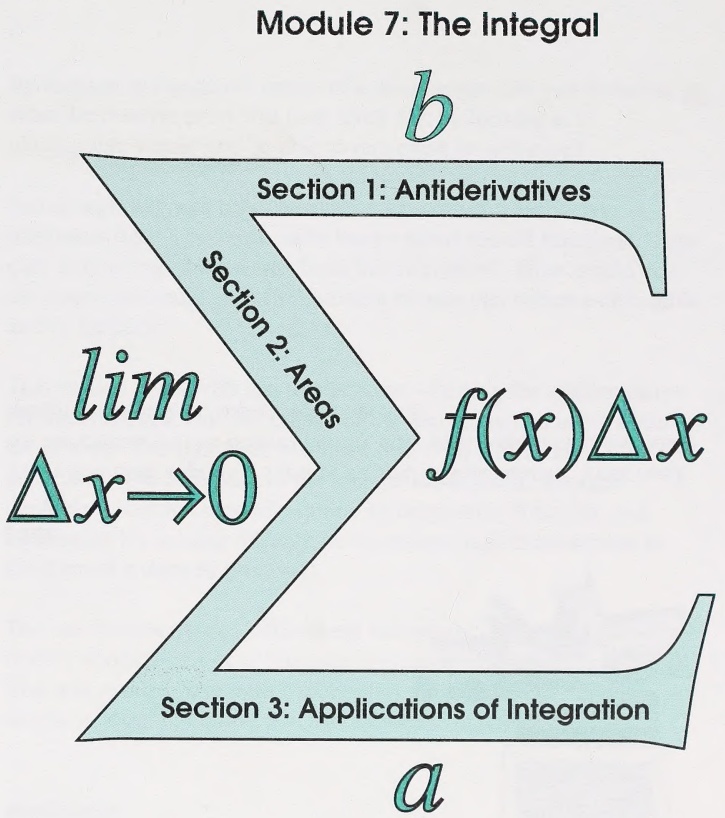
Module Overview

A mathematician and a colleague are walking in an Edmonton neighbourhood. They watch a car speed down the street and then decelerate to a stop at the intersection; they hear a child practising the violin; and they observe two neighbours discussing how much concrete is required to lay a curved driveway. The mathematician comments that these events all relate to calculus and its branches.

Calculus is divided into two main branches: differential calculus and integral calculus. In preceding modules, you worked with differential calculus—the procedures and applications of the derivative. You will now work with integral calculus—the inverse of differentiation. Section 1 deals with the specifics of this process. Throughout this module you will use integral calculus to model real-world situations such as those described in the neighbourhood scene.

Section 2 introduces you to finding area, a basic application of integral calculus. Later in this section, you will use areas to link differential and integral calculus through the Fundamental Theorem of Calculus.

In Section 3, you will use the connection between derivatives and integrals to relate position, displacement, velocity, and acceleration. Then you will apply those relationships to harmonic motion (such as the vibrations of the violin strings).



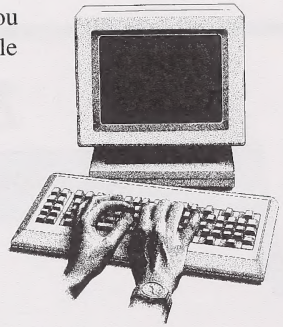
Evaluation

Your mark for this module will be determined by how well you complete the assignments at the end of each section and at the end of the module. In this module you must complete three section assignments and one final module assignment. The mark distribution is as follows:

Section 1 Assignment	**36 marks**
Section 2 Assignment	**40 marks**
Section 3 Assignment	**14 marks**
Final Module Assignment	**10 marks**
Total	**100 marks**

When doing the assignments, work slowly and carefully. You must do each assignment independently; but if you are having difficulties, you may review the appropriate section in this module booklet.

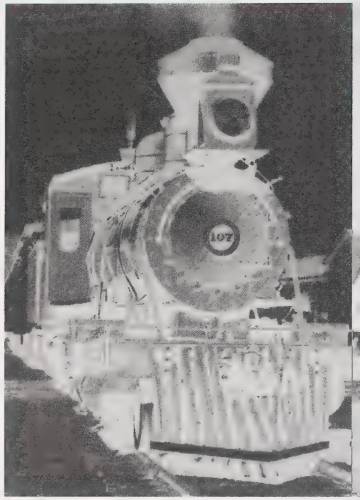
If you are working on a CML terminal, you will have a module test as well as a module assignment.



Note

There is a final supervised test at the end of this course. Your mark for the course will be determined by how well you do on the module assignments and the supervised final test.

Section 1: Antiderivatives



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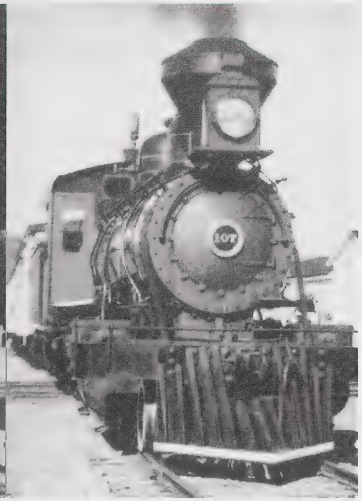

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By looking at a negative image of a photograph, can you visualize what the positive print will look like? Or, by looking at a photograph, would you be able to recognize its negative?

Just as a photograph originates from a negative, a derivative originates from a function, or at least a set of related functions. How easy is it to work backwards from the derivative? How would you use your knowledge of differentiation to help you match a derivative and its function?

This section deals with just this process—finding the antiderivative (or integral), and how the graphs of the derivative and antiderivative are related. The first three activities deal with the basics of this procedure: the definition of the antiderivative itself, why the derivative does not uniquely define an originating function, and techniques for solving differential equations (equations written in the form of a derived function).

The last three activities refine these techniques. The basic rules for finding antiderivatives or integrals are summarized and expanded. You will practice integration techniques on polynomial functions, simple rational functions, and trigonometric functions.

Activity 1: Antiderivatives

In winter, the snow cover makes it easy to retrace your steps or to determine where animal tracks originated. In calculus, it should be possible to work backwards from the derivative to the original function. What is differentiation's inverse operation?

Example 1

If $f'(x) = 3x^2$, what is $f(x)$?

Solution

You may recognize $f'(x) = 3x^2$ as the derivative of $f(x) = x^3$. However, is this function the only possibility? Other functions with the same derivatve are $y = x^3 + 7$, $y = x^3 - 1$, $y = x^3 + \pi$, and so on. In fact, any function of the form $y = x^3 + C$ will do, where C is an arbitrary constant (since the derivative of a constant is 0).

The original function from which the derivative was obtained is termed the **antiderivative**, **integral**, or **primitive**. Antidifferentiation is the **inverse operation** of differentiation. Like the derivative, the antiderivative may be represented in various ways.

If $y' = f(x)$ is the derived function, then $y = F(x)$ is used to denote the antiderivative. Remember that any function of the form $y = F(x) + C$, where C is an arbitrary constant, will have the same derivative. This is the **general antiderivative**.

$$\frac{d}{dx}[F(x) + C] = f(x)$$

The antiderivative, or integral, is commonly represented by the following:

$$\int f(x)\, dx$$

This is read as **the integral of $f(x)$ with respect to x**. The symbol $\int \ldots dx$ is the inverse of the symbol $\frac{d}{dx}\ldots$. In Example 1, the solution may have been written as follows:

$$\int 3x^2\, dx = x^3 + C$$

This is read as **the integral of $3x^2$ is $x^3 + C$**.

This integral is referred to as the **indefinite integral**, since the **constant of integration C** is arbitrary.

The symbol $\int$ is an old-fashioned letter *S*. The reason the letter *S* is used for the integral will become apparent in Section 2.

The two notations in the following statement are equivalent. The indefinite integral is the general antiderivative.

$$\int f(x)\,dx = F(x) + C$$

Example 2

Find the general antiderivative of $f(x) = 2x + 3$.

Solution

Because $2x$ is the derivative of x^2 and 3 is the derivative of $3x$, the general antiderivative is $F(x) = x^2 + 3x + C$.

To check your answer, simply differentiate.

$$\begin{aligned}\frac{d}{dx}[F(x)] &= \frac{d}{dx}\left(x^2 + 3x + C\right)\\ &= 2x + 3 + 0\\ &= 2x + 3\\ &= f(x)\end{aligned}$$

Note: This question could have been posed as "Integrate $\int (2x+3)\,dx$." The solution is as before.

$$\int (2x+3)\,dx = x^2 + 3x + C$$

Now try the following questions.

1. Find the general antiderivative of $f(x) = 1$.

2. Find the indefinite integral of $\int x\,dx$.

3. Integrate $\int x^2\,dx$.

4. Find the general primitive of $f(x) = x^3$.

Check your answers by turning to the Appendix.

Did you notice that the solutions were obtained by reversing the power rule? The solution is obtained by increasing the exponent by one and dividing the power by the increased exponent.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \text{ where } n \neq -1.$$

This must be correct since $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^{(n+1)-1}}{n+1}$

$$= x^n$$

This rule will work for all real values of n except -1.

5. Integrate the following:

a. $\int x^{-3}\,dx$ **b.** $\int \sqrt{x}\,dx$

c. $\int \frac{1}{x^4}\,dx$ **d.** $\int \frac{1}{\sqrt[3]{x}}\,dx$

Check your answers by turning to the Appendix.

In this activity you determined how to find a function given only its derivative. Just as in winter, where the snow allows you to follow your steps back to your starting point, there are usually enough "markers" in a given derivative to determine the original function.

Activity 2: Families of Curves

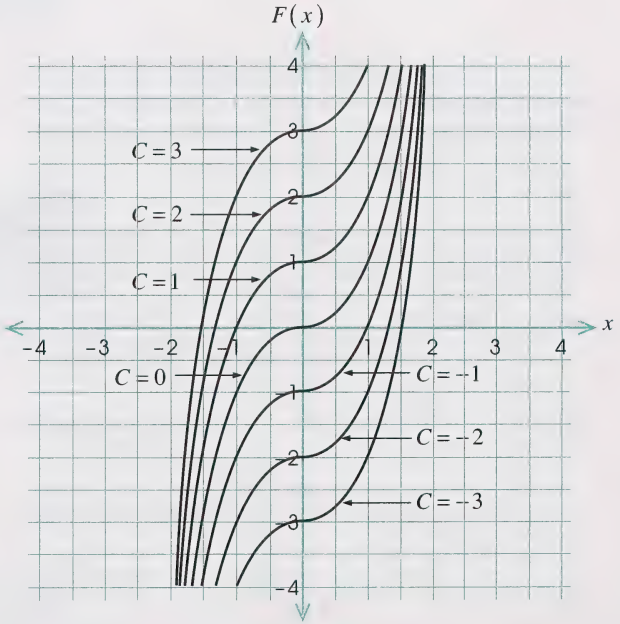
Have you ever noticed that when a glacier is viewed from a distance, its flow lines form long parallel curves? The midportion of the glacier flows faster than the ice along the sides. Material carried by the ice marks the different rates of flow, forming a "family" of curves. In this activity, the antiderivative will be used to generate families of curves.

Example 1

Using various values for the constant C, draw the graph of the general antiderivative for $f(x) = 3x^2$. Compare the resulting curves.

Solution

The general antiderivative is $F(x) = x^3 + C$.



The general primitive $F(x) = x^3 + C$ can be viewed as representing a **family of curves**. Each curve is the graph of $y = x^3$ translated vertically C units. In the previous graph, the curves are **parallel**. The slopes of the tangent lines would be the same for all of these curves for a given value of x. For instance, when $x = 1$, the slope of the tangent to any of these curves is $f(1) = 3(1)^2 = 3$.

1. Draw the graphs of the families of curves for which each of the following functions represents a derivative.

a. $f(x) = 2x + 1$ **b.** $f(x) = 3$

c. $f(x) = \sqrt{x}$ **d.** $f(x) = 1 - \frac{1}{x^2}$

Check your answers by turning to the Appendix.

As you have seen, a function is not uniquely defined from its derivative alone. In the next example, you will be given additional information which will allow you to find a particular curve.

Example 2

Find the equation of the curve with a derivative $(2x - 3)$ and which passes through $(1, 2)$. Draw the graph.

Solution

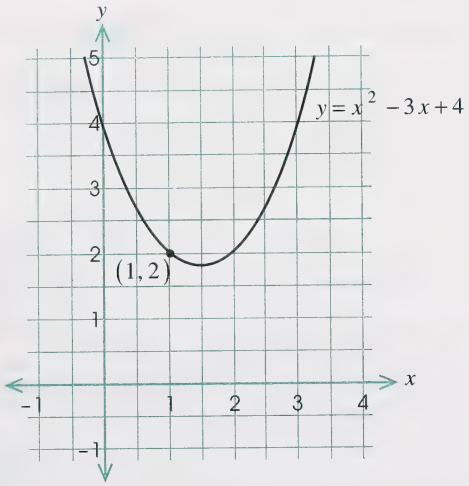
Begin by finding the antiderivative.

$$y = \int (2x - 3)\, dx$$
$$= x^2 - 3x + C$$

The indefinite integral represents a family of curves. Only one of those curves passes through $(1, 2)$.

Substitute $x = 1$ and $y = 2$ into the equation; then evaluate C.

$$2 = 1^2 - 3(1) + C$$
$$2 = 1 - 3 + C$$
$$C = 4$$
$$\therefore\ y = x^2 - 3x + 4$$



2. Find the equation of each curve given the slope of the curve and a point on the curve; then draw the graphs of each curve.

a. $\frac{dy}{dx} = 5;\ (3, 7)$

b. $\frac{dy}{dx} = x^3 - 2x^2 - 1;\ (0, 1)$

Check your answers by turning to the Appendix.

The constant of integration C produces a family of parallel curves. Some families of curves look a lot like the flow lines of the glacier in the photograph at the beginning of Activity 2.

Activity 3: Differential Equations

With practice, Daniel has become a respected player on his team. He has learned to anticipate the behaviour of the puck. He knows its motion depends on the force of the shot and on the ice surface. The forces that act on the puck affect its position, velocity, and acceleration.

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Recall that **velocity** and **acceleration** are derivatives of the **position function**. Therefore, equations describing motion may contain derivatives.

A **differential equation** is an equation which contains derivatives. In this section you will solve differential equations which consist of no more than two variables. The derivatives will be of one of those variables with respect to the other. The derivatives may be first-order derivatives or higher-order derivatives. For instance, $\frac{dy}{dx} + 3xy = 0$.

Differential equations are used to model numerous situations in the sciences (such as growth and decay, motion problems, heating and cooling, electromagnetic fields, and gravitation). In this activity, you will be solving first-order differential equations of the form $\frac{dy}{dx} = f(x)$ by determining the antiderivative $y = F(x)$. A function $y = F(x)$ is a solution of the differential equation, if $y = F(x)$ is differentiable and $\frac{d}{dx}F(x) = f(x)$.

Example 1

Solve $\frac{dy}{dx} + 2x = 0$.

Solution

Isolate the derivative.

$$\frac{dy}{dx} = -2x$$

Determine the antiderivative.

$$y = \int (-2x)\, dx$$
$$= -x^2 + C$$

In Example 1, the solution is a general one. To obtain a particular solution, additional conditions on the differential equation must be imposed. This is similar to what you studied in Activity 2, where you were asked to find a particular member of a family of curves.

For the differential equation $\frac{dy}{dx} = f(x)$, if you are told that $y = y_0$ when $x = x_0$, the constant of integration may be determined. The condition imposed is called an **initial condition**.

Example 2

From just inside the blueline, Daniel shoots a puck towards the net. The velocity v (in metres per second) at time t (in seconds) of the puck moving along the ice surface is given by $v = -2t + 30$. If the position s (in metres), measured from the blueline, is $s = 2$ when $t = 0$, determine the position s of the puck at any time t.

Solution

Since $v = \frac{ds}{dt}$, then $v = -2t + 30$ may be written as the differential equation $\frac{ds}{dt} = -2t + 30$.

Find the antiderivative.

$$s = \int (-2t + 30)\, dt$$
$$= -t^2 + 30t + C$$

The constant of integration C can now be determined from the initial condition, $s = 2$ when $t = 0$. This condition is called an initial condition, as it defines the **initial position** of the puck.

$$2 = -0^2 + 30(0) + C$$
$$C = 2$$

Therefore, the position at any time is $s = -t^2 + 30t + 2$.

1. Find the most general solution to each differential equation.

a. $\frac{dy}{dx} = 2x^{\frac{1}{3}} + 4x + 2$

b. $\frac{dy}{dx} = (3x - 1)^2$

c. $\frac{dv}{dt} = -10t + 5$

2. Solve each differential equation subject to the given initial conditions.

   a. $\frac{ds}{dt} = 2t + 4;\ s = 5$ when $t = 0$

   b. $\frac{dy}{dx} = \frac{2}{x^2};\ y = 16$ when $x = 2$

   c. $\frac{dy}{dx} - 3x + 2 = 0;\ y = 4$ when $x = 1$

Check your answers by turning to the Appendix.

In this module and in Module 8, you will use first-order and higher-order differential equations to model displacement, velocity, and acceleration problems.

Activity 4: Basic Integration Techniques

In the preceding activities you used a number of integration techniques that will now be formally stated. In the following formulas, F and G represent differentiable functions, f and g are their derivatives, and a and C are constants.

Formula 1: $\int f(x)\,dx = F(x) + C$

Formula 2: $\int af(x)\,dx = a\int f(x)\,dx$

Formula 2 states that the constant a may be moved across the integral sign. It does not say that a variable can be moved across the integral sign. This formula can be proven as follows:

$$\begin{aligned} a\int f(x)\,dx &= a\left[F(x) + C\right] \\ &= aF(x) + aC \end{aligned}$$

The derivative of this expression is

$$\begin{aligned} \tfrac{d}{dx}\left[aF(x) + aC\right] &= \tfrac{d}{dx}\left[aF(x)\right] + 0 \\ &= a\tfrac{d}{dx}F(x) \\ &= af(x) \end{aligned}$$

This is simply the derivative of the $\int af(x)dx$. Therefore, Formula 2 is true.

Formula 3

$$\int\left[f(x) \pm g(x)\right]dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Formula 3 states that the integral of a sum or difference is the sum or difference of the integrals.

This rule may be verified as follows:

$$\int f(x)\,dx \pm \int g(x)\,dx = F(x) \pm G(x) + C$$

Differentiate this expression.

$$\frac{d}{dx}\left[F(x) \pm G(x) + C\right] = \frac{d}{dx}F(x) \pm \frac{d}{dx}G(x) + 0$$
$$= f(x) + g(x)$$

This is simply the derivative of $\int [f(x) \pm g(x)]dx$. Therefore, Formula 3 is true.

Formula 4: $\int x^n\,dx = \frac{x^{n+1}}{n+1}$, where $n \neq -1$

Now you will apply these rules to polynomials and simple algebraic functions.

Example 1

Find the indefinite integral $\int \left(2x^2 - 5x - 2x^{-3}\right)\,dx$.

Solution

$$\int \left(2x^2 - 5x - 2x^{-3}\right)\,dx = \int 2x^2\,dx - \int 5x\,dx - \int 2x^{-3}\,dx$$
$$= 2\int x^2\,dx - 5\int x\,dx - 2\int x^{-3}\,dx$$
$$= 2\frac{x^3}{3} - 5\frac{x^2}{2} - 2\frac{x^{-2}}{-2} + C$$
$$= \frac{2}{3}x^3 - \frac{5}{2}x^2 + x^{-2} + C$$

Example 2

Find the indefinite integral $\int \left(\sqrt{x} + 1\right)^2\,dx$.

Solution

Begin by expanding the power.

$$\int \left(\sqrt{x} + 1\right)^2\,dx = \int \left(x + 2\sqrt{x} + 1\right)\,dx$$
$$= \int x\,dx + 2\int \sqrt{x}\,dx + \int 1\,dx$$
$$= \int x^1\,dx + 2\int x^{\frac{1}{2}}\,dx + \int 1\,dx \quad \text{(Express as powers.)}$$
$$= \frac{x^2}{2} + 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + x + C$$
$$= \frac{x^2}{2} + \frac{4x^{\frac{3}{2}}}{3} + x + C$$

Find the indefinite integral in each case.

1. $\int \left(5x^2 - 3\right)^2 \, dx$

2. $\int \left(x^{\frac{1}{3}} - 4x + 1 \right) dx$

3. $\int \left(\frac{1}{3x^4} - 6x^4 \right)^2 dx$

4. $\int \left(\frac{2}{\sqrt[3]{x}} + \sqrt{x} \right) dx$

Check your answers by turning to the Appendix.

Remember, you can always check your answers by differentiating. Integration and differentiation are inverse operations.

Activity 5: Comparison Techniques

A technique in problem solving is to compare the problem you want to solve with a problem the solution of which is known or can be readily obtained.

In this activity you will use the chain rule and **comparison techniques** to assist you to find indefinite integrals such as $\int 2x\left(3x^2 - 5\right)^7 \, dx$.

Obviously, expanding this expression and integrating term by term would be extremely time-consuming!

Recall that the chain rule for powers is

$\frac{d}{dx}\left[f(x)\right]^n = n\left[f(x)\right]^{n-1} f'(x)$.

Example 1

Integrate $\int 2x\left(3x^2 - 5\right)^7 \, dx$.

Solution

Use comparison techniques. If the question were simply $\int x^7 \, dx$, the solution would be $\frac{x^8}{8} + C$. What if the solution to this problem were $y = \frac{a\left(3x^2 - 5\right)^8}{8} + C$? Then the derivative of that function should be $2x\left(3x^2 - 5\right)^7$.

But what is the derivative?

Apply the chain rule.

Let $f(x) = 3x^2 - 5$. Now $f'(x) = 6x$.

$$\therefore \frac{dy}{dx} = \frac{a}{8}(8)\left[f(x)\right]^7 f'(x)$$

$$= a\left(3x^2 - 5\right)^7 (6x)$$

$$= 6ax\left(3x^2 - 5\right)^7$$

But you know $\frac{dy}{dx} = 2x\left(3x^2 - 5\right)^7$.

Compare the two expressions and solve for a.

$$6ax\left(3x^2 - 5\right)^7 = 2x\left(3x^2 - 5\right)^7$$
$$6a = 2$$
$$a = \frac{1}{3}$$

Substitute $a = \frac{1}{3}$ into $y = \frac{a\left(3x^2 - 5\right)^8}{8} + C$.

$$y = \frac{\frac{1}{3}\left(3x^2 - 5\right)^8}{8} + C$$
$$= \frac{\left(3x^2 - 5\right)^8}{24} + C$$

$$\therefore \int 2x\left(3x^2 - 5\right)^7 \, dx = \frac{\left(3x^2 - 5\right)^8}{24} + C$$

Check your answer by differentiating. In this case (Example 1), you obtain $y' = 2x\left(3x^2 - 5\right)^7$.

Example 2

Integrate $\int 6(3x-1)^{\frac{3}{2}} \, dx$.

Solution

If the question were simply $\int x^{\frac{3}{2}} \, dx$, the answer would be $\frac{2}{5}x^{\frac{5}{2}} + C$. Assume then that the answer is of the form $y = \frac{2}{5}a(3x-1)^{\frac{5}{2}} + C$. If this function is the integral, then its derivative must equal $6(3x-1)^{\frac{3}{2}}$. Now the derivative of $y = \frac{2}{5}a(3x-1)^{\frac{5}{2}} + C$ can be obtained by applying the chain rule.

$$\frac{dy}{dx} = \frac{2}{5}a\left(\frac{5}{2}\right)(3x-1)^{\frac{3}{2}}(3)$$
$$= 3a(3x-1)^{\frac{3}{2}}$$

Compare this result with $6(3x-1)^{\frac{3}{2}}$, and solve for a.

$$3a(3x-1)^{\frac{3}{2}} = 6(3x-1)^{\frac{3}{2}}$$
$$3a = 6$$
$$a = 2$$

Substitute $a = 2$ into $y = \frac{2}{5}a(3x-1)^{\frac{5}{2}} + C$.

$$y = \frac{2}{5}(2)(3x-1)^{\frac{5}{2}} + C$$
$$= \frac{4}{5}(3x-1)^{\frac{5}{2}} + C$$

$$\therefore \int 6(3x-1)^{\frac{3}{2}}\,dx = \frac{4}{5}(3x-1)^{\frac{5}{2}} + C$$

Remember: Integration and differentiation are inverse operations. In this activity, it is wise to confirm your answers by differentiating.

In Example 2, the solution was $y = \frac{4}{5}(3x-1)^{\frac{5}{2}} + C$. Its derivative is as follows:

$$\frac{dy}{dx} = \frac{4}{5}\left(\frac{5}{2}\right)(3x-1)^{\frac{5}{2}-1}(3)$$
$$= 6(3x-1)^{\frac{3}{2}}$$

The derivative must be the same as the **integrand**—the expression being integrated in the original question.

Example 3

Integrate $\int x\left(x^2-1\right)^{-2}\,dx$.

Solution

If the question read $\int x^{-2}\,dx$, the answer would be $-x^{-1} + C$.

Therefore, assume that the answer is of the form $y = -a\left(x^2-1\right)^{-1}$. If this function is the integral, then its derivative must equal $x\left(x^2-1\right)^{-2}$. Now the derivative of $y = -a\left(x^2-1\right)^{-1} + C$ can be obtained by applying the chain rule.

$$\frac{dy}{dx} = -a(-1)\left(x^2-1\right)^{-2}(2x)$$
$$= 2ax\left(x^2-1\right)^{-2}$$

Compare this result with $x\left(x^2-1\right)^{-2}$, and solve for a.

$$2ax\left(x^2-1\right)^{-2} = x\left(x^2-1\right)^{-2}$$
$$2a = 1$$
$$a = \frac{1}{2}$$

Substitute $a = \frac{1}{2}$ into $y = -a\left(x^2 - 1\right)^{-1} + C$

$$= -\frac{1}{2}\left(x^2 - 1\right)^{-1} + C$$

$$\therefore \int x\left(x^2 - 1\right)^{-2}\, dx = -\frac{1}{2}\left(x^2 - 1\right)^{-1} + C$$

1. Find the indefinite integral for each of the following:

a. $\int 2x\left(x^2 + 1\right)^3\, dx$

b. $\int (1 - x)^{\frac{1}{2}}\, dx$

c. $\int 6x^2\left(x^3 - 1\right)^{-2}\, dx$

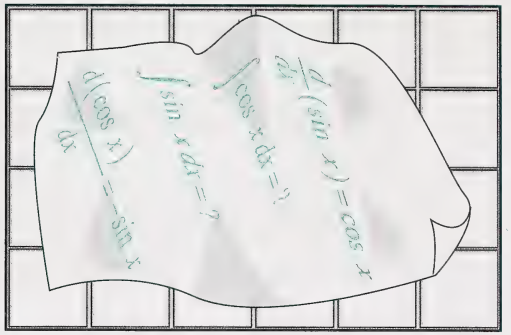
d. $\int \dfrac{3x}{\left(x^2 + 1\right)^3}\, dx$

2. Find the indefinite integral for $\int (x - 1)^2\, dx$ by using the comparison techniques; then find the indefinite integral by expanding the binomial first and integrating term by term. Are the answers to each method equivalent?

Check your answers by turning to the Appendix.

Comparison techniques are really guess-and-test procedures. The test of your answer is its derivative. The derivative must be the integrand.

Activity 6: Trigonometric Functions



Now that you know integration and differentiation are inverse operations, many of the integral formulas can be obtained from a list of derivatives. Writing out the derivative formulas and preparing lists of integrals is an effective way to review.

In this activity you will reverse the derivative formulas for trigonometric functions to obtain antiderivatives.

Recall the following derivative formulas for trigonometric functions, covered in Module 4.

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\csc x) = -\csc x \cot x$

From the derivative formulas for trigonometric functions, the six new integration formulas are as follows:

- $\int \cos\ x\,dx = \sin\ x + C$
- $\int \sin\ x\,dx = -\cos\ x + C$
- $\int \sec^2\ x\,dx = \tan\ x + C$
- $\int \csc^2\ x\,dx = -\cot\ x + C$
- $\int \sec\ x\ \tan\ x\,dx = \sec\ x + C$
- $\int \csc\ x\ \cot\ x\,dx = -\csc\ x + C$

You will now apply these formulas and the integration techniques presented previously in this section.

Example 1

Integrate $\int \cos\ 4x\,dx$.

Solution

Because $\int \cos\ x\,dx = \sin\ x + C$, the solution to $\int \cos\ 4x\,dx$ must be of the form $a \sin\ 4x$ for some coefficient a. To find that coefficient, compare the derivative of $a \sin\ 4x$ with the integrand $\cos\ 4x$.

$$\frac{d}{dx}(a \sin\ 4x) = a(\cos\ 4x)(4)$$
$$= 4a \cos\ 4x$$

Remember to apply the chain rule and differentiate $4x$.

Now $4a \cos\ 4x = \cos\ 4x$

$$4a = 1$$
$$a = \frac{1}{4}$$

$$\therefore \int \cos\ 4x\,dx = \frac{1}{4} \sin\ 4x + C$$

Remember: The antiderivative may be asked for in a variety of ways.

Example 2

Solve $\frac{dy}{dx} = 5 \sin\ 3x\,dx$.

Solution

Since $\int \sin\ x\,dx = -\cos\ x + C$, assume that your solution is of the form $y = -a \cos\ 3x + C$ for some coefficient a.

$$\frac{dy}{dx} = -a(-\sin\ 3x)(3)$$
$$= 3a \sin\ 3x$$

Compare the derivative with $5 \sin 3x$.

$$3a \sin 3x = 5 \sin 3x$$
$$3a = 5$$
$$a = \frac{5}{3}$$

$$\therefore\ y = -\frac{5}{3} \cos 3x + C$$

As you see, comparison techniques are an important part of the integration process for trigonometric functions.

Example 3

Determine the family of curves with the derivative $\frac{dy}{dx} = x \sin \left(2x^2\right) + 1$. Graph representative curves using a graphing calculator (or computer program).

Solution

$$y = \int x \sin \left(2x^2\right)\, dx + \int 1\, dx$$

Since $\int \sin x\, dx = -\cos x + C$, assume that the solution to the first integral is of the form $-a \cos \left(2x^2\right) + C$ for some coefficient a.

$$\therefore\ \frac{d}{dx}\left[-a \cos \left(2x^2\right) + C\right] = -a\left[-\sin \left(2x^2\right)\right](4x)$$
$$= 4ax \sin \left(2x^2\right)$$

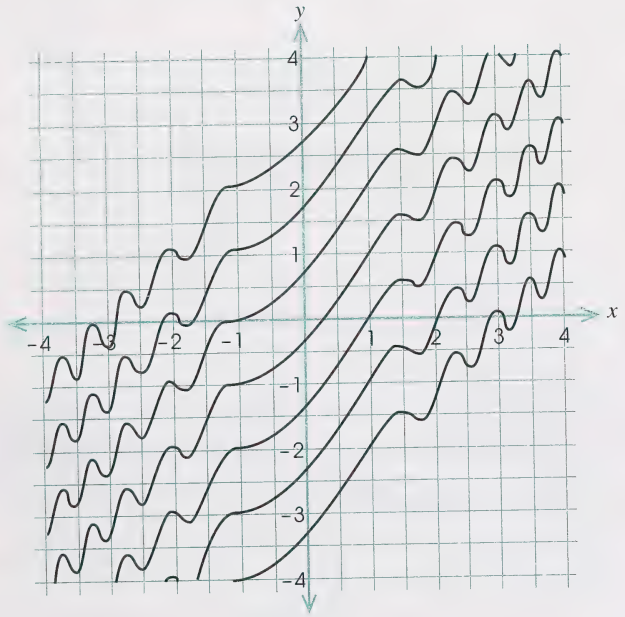
Compare this result with the integrand $x \sin \left(2x^2\right)$.

$$4ax \sin \left(2x^2\right) = x \sin \left(2x^2\right)$$
$$4a = 1$$
$$a = \frac{1}{4}$$

$$\therefore\ \int x \sin \left(2x^2\right)\, dx = -\frac{1}{4} \cos \left(2x^2\right) + C$$

Therefore, the family of curves is represented by the equation $y = -\frac{1}{4} \cos \left(2x^2\right) + x + C$.

The graph is as follows:



Example 4

Find the general antiderivative of $f(x) = 5\sin^3 x \cos x$.

Solution

Assume the antiderivative is of the form $F(x) = \frac{a\sin^4 x}{4} + C$.

$$\therefore \frac{d}{dx}[F(x)] = a\frac{(4\sin^3 x)(\cos x)}{4}$$

$$= a\sin^3 x \cos x$$

Remember to think of $\sin^4 x$ as $(\sin x)^4$, and apply the chain rule.

Also, $\frac{d}{dx}[F(x)] = f(x) = 5\sin^3 x \cos x$.

$$\therefore a\sin^3 x \cos x = 5\sin^3 x \cos x$$

$$a = 5$$

$$\therefore F(x) = \frac{5}{4}\sin^4 x + C.$$

Now apply these techniques in the following questions.

1. Find the general primitive of $f(x) = \sin(3x - 1)$.

2. Integrate $\int 3\sec 2x \tan 2x\, dx$.

3. Solve $\frac{dy}{dx} = \csc^2 3x\, dx$.

4. Find the equation of the family of curves for which $y' = 2 \sin x \cos x$. First express the antiderivative in terms of $\sin x$; then in terms of $\cos x$. Show that the answers are equivalent.

5. Integrate $\int 5 \sin^3 (4x+1) \cos (4x+1)\, dx$.

6. Solve $\frac{dy}{dx} = x^2 \cos \left(x^3 + 1\right)$ if $y = 1$ when $x = -1$.

7. Find the general solution to $\frac{dy}{dx} = \frac{\sin x}{\cos^3 x}$.

Check your answers by turning to the Appendix.

Memorize the integrals and derivatives of each trigonometric function. You will save a lot of time in your future work!

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

You may have noticed patterns in the antiderivatives of trigonometric expressions of the form $a \sin bx$ or $a \cos bx$, where a and b are real non-zero constants.

First, $\int a \sin bx\, dx = -\frac{a}{b} \cos bx + C$. This formula may be checked by differentiating.

$$\begin{aligned} \frac{dy}{dx} &= -\frac{a}{b}(-\sin bx)(b) \\ &= a \sin bx \end{aligned}$$

Second, $\int a \cos bx\, dx = \frac{a}{b} \sin bx + C$. Again, check by differentiating.

$$\begin{aligned} \frac{dy}{dx} &= \frac{a}{b}(\cos bx)(b) \\ &= a \cos bx \end{aligned}$$

Example 1

Integrate $\int 2 \sin 7x\, dx$.

Solution

Because $\int a \sin bx\, dx = -\frac{a}{b} \cos bx + C$, and $a = 2$ and $b = 7$,

$\int 2 \sin 7x\, dx = -\frac{2}{7} \cos 7x + C$.

Example 2

Solve $\frac{dy}{dx} = -3 \cos \frac{x}{3}$.

Solution

Recall $\int a \cos bx \, dx = \frac{a}{b} \sin bx + C$. In this example, $a = -3$ and $b = \frac{1}{3}$.

$$\therefore y = -\frac{3}{\frac{1}{3}} \sin \frac{x}{3} + C$$

$$= -9 \sin \frac{x}{3} + C$$

Evaluate each of the following integrals.

1. $\int 6 \cos 7x \, dx$

2. $\int 2 \sin (-3x) \, dx$

3. $\int 5 \cos \frac{x}{5} \, dx$

4. $\int \frac{1}{3} \sin \left(-\frac{x}{3}\right) dx$

Check your answers by turning to the Appendix.

Enrichment

Integration by parts is a general technique of integration. Based on the derivative of a product, this technique is used to express an integral that is difficult to evaluate in terms of a second integral that is easier to evaluate.

The procedure for integration by parts is developed as follows:

Step 1: Assume f and g are differentiable functions.

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)g'(x) + g(x)f'(x)$$

Or, equivalently,

$$f(x)g'(x) = \frac{d}{dx}[f(x) \cdot g(x)] - g(x)f'(x)$$

Step 2: Integrate both sides of the equation. Thus, the procedure for integration by parts is given.

$$\int f(x)g'(x) \, dx = f(x)g(x) - \int g(x)f'(x) \, dx + C$$

In the examples that follow, you will discover that it is only by careful selection of f and g in the original problem that the integral $\int g(x)f'(x) \, dx$ is, in fact, easier to evaluate than the original integral $\int f(x)g'(x) \, dx$.

Example 1

Evaluate $\int x \sin\ x\, dx$.

Solution

This problem cannot be approached by the normal comparison techniques outlined in Section 1, Activity 6. However, you can integrate by parts. Decide which of x and $\sin\ x$ will be designated as $f(x)$ and which will be designated as $g'(x)$.

Let $f(x) = x$ and $g'(x) = \sin\ x$.

$\therefore f'(x) = 1$ and $g(x) = -\cos\ x$

Substitute into

$$\int f(x)g'(x)\ dx = f(x)g(x) - \int g(x)f'(x)\ dx + C.$$

$$\begin{aligned}\int x \sin\ x\, dx &= x(-\cos\ x) - \int(-\cos\ x)(1)\ dx + C \\ &= -x \cos\ x + \int \cos\ x\, dx + C \\ &= -x \cos\ x + \sin\ x + C\end{aligned}$$

To verify, differentiate.

$$\begin{aligned}\frac{d}{dx}(-x \cos\ x + \sin\ x + C) &= \frac{d}{dx}(-x \cos\ x) + \frac{d}{dx}(\sin\ x) + 0 \\ &= -x\frac{d}{dx}(\cos\ x) + \cos\ x\frac{d}{dx}(-x) + \cos\ x \\ &= -x(-\sin\ x) + (\cos\ x)(-1) + \cos\ x \\ &= x \sin\ x\end{aligned}$$

Example 2

Evaluate $\int x^2 \cos\ x\, dx$.

Solution

To apply integration by parts, you must decide which of x^2 and $\cos\ x$ will be designated as $f(x)$ and which will be designated as $g'(x)$.

Let $f(x) = x^2$ and $g'(x) = \cos\ x$.

Therefore, $f'(x) = 2x$ and $g(x) = \sin\ x$.

Substitute into

$$\int f(x)g'(x)\ dx = f(x)g(x) - \int g(x)f'(x)\ dx + C.$$

$$\begin{aligned}\int x^2 \cos\ x\, dx &= x^2 \sin\ x - \int 2x \sin\ x\, dx + C \\ &= x^2 \sin\ x - 2\int x \sin\ x\, dx + C\end{aligned}$$

To evaluate the second integral, integration by parts must be applied a second time.

However, from Example 1, $\int x \sin x\, dx = -x \cos x + \sin x$.

$$\therefore \int x^2 \cos x\, dx = x^2 \sin x - 2(-x \cos x + \sin x) + C$$
$$= x^2 \sin x + 2x \cos x - 2 \sin x + C$$

Now try the following questions.

1. Show that if you let $f(x) = \sin x$ and $g'(x) = x$ in Example 1, the second integral would not have been simpler to evaluate than the first.

2. Integrate $\int x \cos x\, dx$ by parts.

3. Evaluate $\int (x+1) \sin 2x\, dx$.

Check your answers by turning to the Appendix.

Conclusion

In this section you discovered that differentiation has an inverse operation. You found the antiderivatives of polynomials, of rational algebraic functions (by comparison techniques), and of simple trigonometric functions. You identified the indefinite integral $\int f(x)\, dx$ as the sum of an antiderivative $F(x)$ and a constant C.

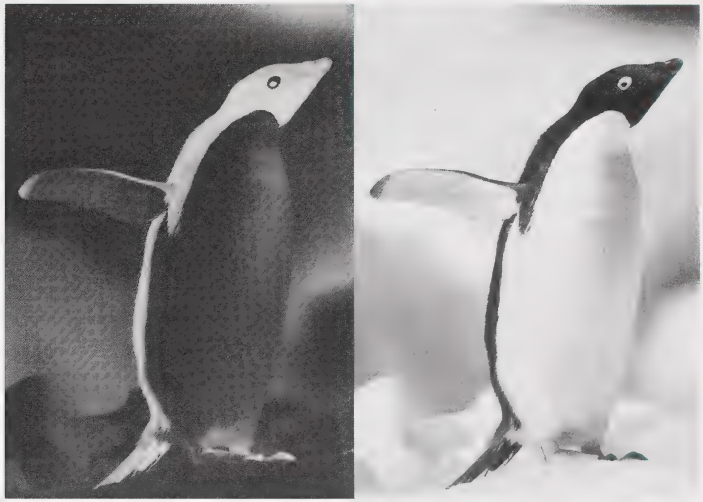
Also, you discovered many different functions have the same derivative. You found the family of curves in which the first derivative was given; you graphed, on the same grid, that family as a sequence of functions. You solved first-order differential equations for general and specific solutions.

In addition, you applied the following general rules for finding integrals for functions of all types. Such types include the following:

- $\int f(x)\, dx = F(x) + C$
- $\int a f(x)\, dx = a \int f(x)\, dx$
- $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$
- $\int x^n\, dx = \frac{x^{n+1}}{n+1}$, where $n \neq -1$

You also studied the following specific integrals for trigonometric functions.

- $\int \cos\ x\ dx = \sin\ x + C$
- $\int \sin\ x\ dx = -\cos\ x + C$
- $\int \sec^2\ x\ dx = \tan\ x + C$
- $\int \csc^2\ x\ dx = -\cot\ x + C$
- $\int \sec\ x\ \tan\ x\ dx = \sec\ x + C$
- $\int \csc\ x\ \cot\ x\ dx = -\csc\ x + C$



Knowing that differentiation and integration are inverse operations, you should now be able to match a derivative and its antiderivative, just as you can match a photograph and its negative. The negative of a black-and-white photograph is the inverse of the positive image, and vice versa—what is light on one is dark on the other.

Just as differentiation reverses integration and integration reverses differentiation, so too the photograph reverses the negative and the negative reverses the photograph.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Section 2: Areas

How many hectares of grain remain to be harvested in the field in the photograph? Farmers are used to estimating areas of such irregular shapes. Is it possible to find the exact area rather than just an approximation? Is there more than one method? Do those methods yield equivalent results?

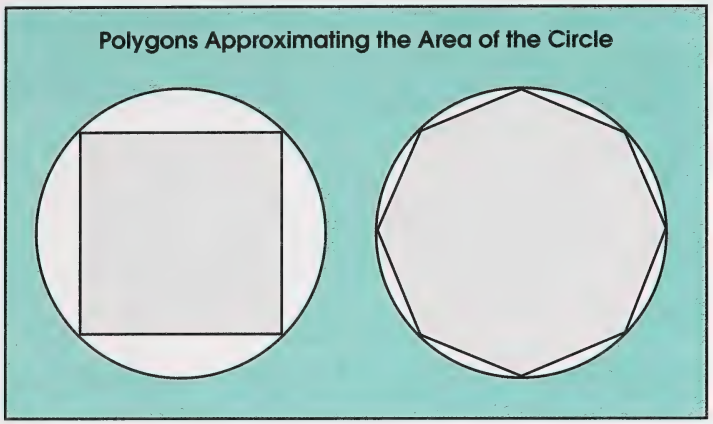
In this section you will investigate how to approximate irregular areas using vertical strips. You will use limits to evaluate the sums of the areas defined by those strips as the widths of those strips decrease.

You will discover that the areas between a curve and the x-axis that were found using limits may also be evaluated using definite integrals. This fact, summarized by the Fundamental Theorem of Calculus, ties differential and integral calculus together.

In the last three activities, you will refine your techniques for evaluating definite integrals, and you will use definite integrals to find more general areas. In particular, you will use the integration techniques, which were covered in Section 1 for polynomial functions, simple rational functions, and trigonometric functions, to find areas between those curves and the x-axis and the areas between two of those curves.

Activity 1: Areas as Limits

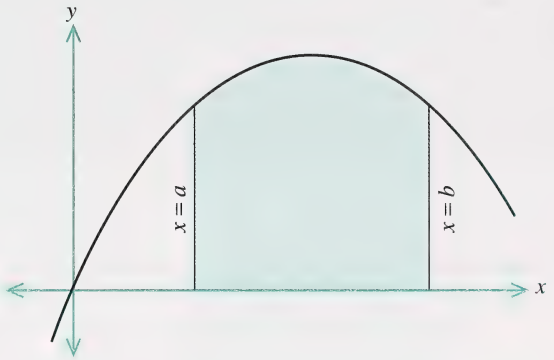
Almost twenty-four centuries ago, Eudoxus, a Greek mathematician and astronomer, invented the **method of exhaustion** for finding areas of shapes such as circles, ellipses, and segments of parabolas. The heart of this method was to fit figures of known area within or around the desired area.



To find the area of a circle by this method, for instance, a polygon would be inscribed in the circle and the area of that polygon calculated. Then the number of sides of the polygon would be systematically increased and the areas of the resulting polygons calculated. As the number of sides increased, a better approximation of the area of the circle occurred; the space between the boundary of the polygon and the circumference of the circle is diminishing or becoming **exhausted**.

This method is closely related to the techniques for finding areas in integral calculus. In this activity you investigate these techniques, and see how they are to be used to define the integral itself.

Look at the problem of calculating the area under a curve between two limiting values, $x = a$ and $x = b$.



The following figure shows such an area that has been divided into six regions of equal width. The area under the curve is the sum of the areas of those six regions.

Figure 1

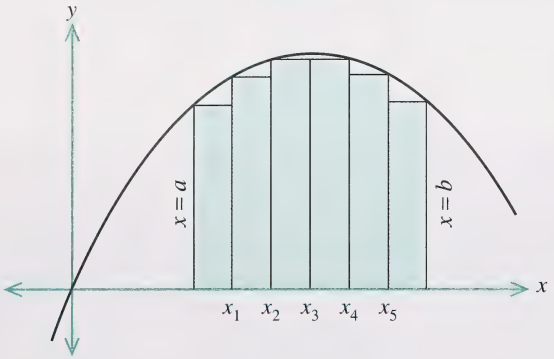


Figure 2

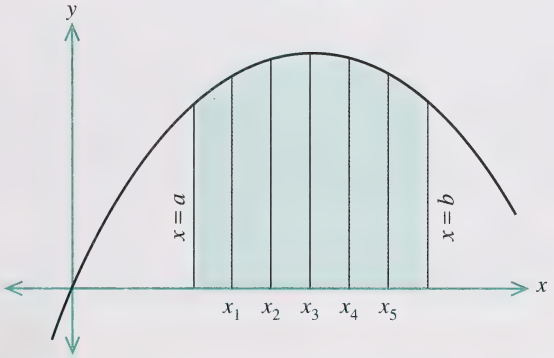
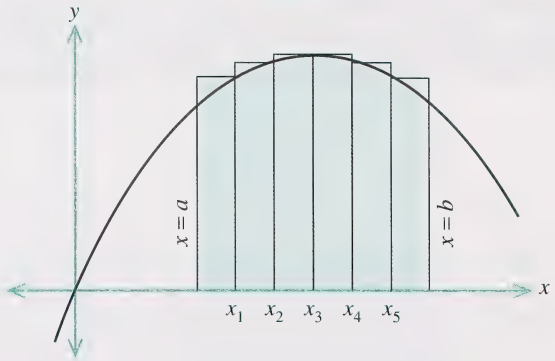


Figure 3



The area of the inscribed rectangles in Figure 1 is less than the area under the curve in Figure 2, which is less than the area of the circumscribed rectangles in Figure 3.

These diagrams show a relationship that will allow you to approximate the area enclosed by the curve, the x-axis, and the lines $x = a$ and $x = b$.

In Figure 1, there is a sequence of six **inscribed** rectangles. In Figure 3, there is a sequence of six **circumscribed** rectangles. By calculating the sum of the areas of the inscribed rectangles and the sum of the areas of the circumscribed rectangles, you arrive at **lower and upper bounds** between which the actual area must lie.

How can you narrow this interval? Can you find the exact value of the area? If you average the areas of the inscribed and circumscribed rectangles, will that give you the area?

Generally, the average of the sums of the areas of the inscribed rectangles and the circumscribed rectangles will not give you the exact area; however, it may give you a good approximation of the area.

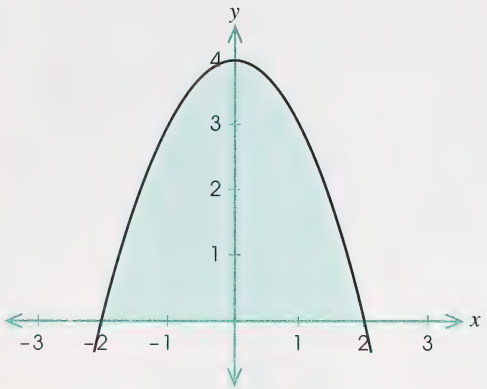
If you decrease the width of your rectangles and increase their number, you will arrive at upper and lower bounds which are closer together; and their average will be a better appoximation of the area under the cruve. Consider the following example.

Example 1

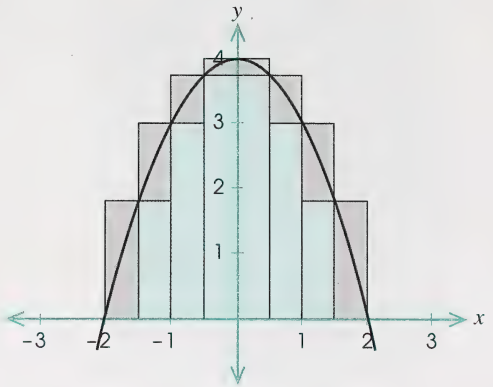
Approximate the area enclosed between the curve defined by $y = 4 - x^2$ and the x-axis.

Solution

First, show the region defined.



Start by dividing the area into eight regions. Construct the sets of inscribed and circumscribed rectangles.



You need to determine the sum of the areas of each set of rectangles. The area of each rectangle is dependent on a y-coordinate and the difference between two successive x-coordinates. In this case, each partition is the same width $(\Delta x = 0.5)$. The y-coordinates can be found using the function defining the curve.

$y = 4 - x^2$

x	y
−2.0	0.00
−1.5	1.75
−1.0	3.00
−0.5	3.75
0.0	4.00
0.5	3.75
1.0	3.00
1.5	1.75
2.0	0.00

For the intervals $-2 \le x \le -1.5$ and $1.5 \le x \le 2$, the inscribed rectangles have heights of 0. However, if you allow these to be rectangles with areas of zero square units, then there will be eight rectangles in total.

Now consider the eight inscribed rectangles.

The areas will be shown in this chart as they appear on the graph from left to right.

Δx	Height	Area
0.5	0.00	0.000
0.5	1.75	0.875
0.5	3.00	1.500
0.5	3.75	1.875
0.5	3.75	1.875
0.5	3.00	1.500
0.5	1.75	0.875
0.5	0.00	0.000
Total Area		8.500

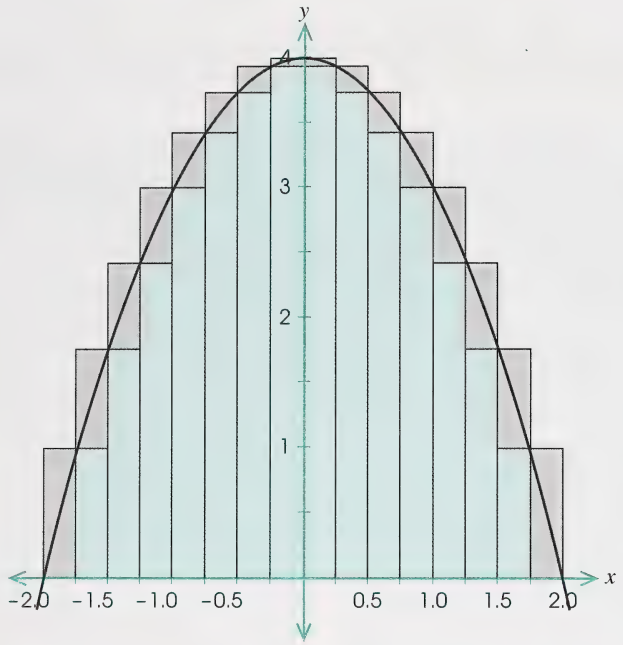
Next, look at the eight circumscribed rectangles.

Δx	Height	Area
0.5	1.75	0.875
0.5	3.00	1.500
0.5	3.75	1.875
0.5	4.00	2.000
0.5	4.00	2.000
0.5	3.75	1.875
0.5	3.00	1.500
0.5	1.75	0.875
Total Area		12.500

From these calculations, you can safely say that the area you are looking for is between 8.5 units^2 and 12.5 units^2.

Thus, you would expect the area to be approximately 10.5 units^2 (the average of the two areas).

What if you used sixteen partitions instead of eight?



$y = 4 - x^2$

x	y
−2.00	0.0000
−1.75	0.9375
−1.50	1.7500
−1.25	2.4375
−1.00	3.0000
−0.75	3.4375
−0.50	3.7500
−0.25	3.9375
0.00	4.0000
0.25	3.9375
0.50	3.7500
0.75	3.4375
1.00	3.0000
1.25	2.4375
1.50	1.7500
1.75	0.9375
2.00	0.0000

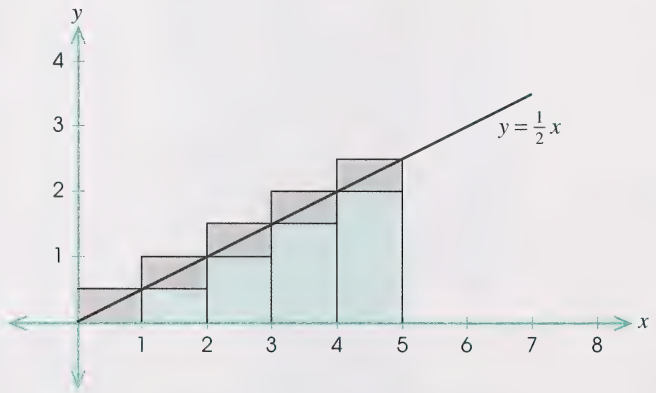
Area of Sixteen Inscribed Rectangles		
Δx	Height	Area
0.25	0.0000	0.000 000
0.25	0.9375	0.234 375
0.25	1.7500	0.437 500
0.25	2.4375	0.609 375
0.25	3.0000	0.750 000
0.25	3.4375	0.859 375
0.25	3.7500	0.937 500
0.25	3.9375	0.984 375
0.25	3.9375	0.984 375
0.25	3.7500	0.937 500
0.25	3.4375	0.859 375
0.25	3.0000	0.750 000
0.25	2.4375	0.609 375
0.25	1.7500	0.437 500
0.25	0.9375	0.234 375
0.25	0.0000	0.000 000
Total Area		9.625 000

Area of Sixteen Circumscribed Rectangles		
Δx	Height	Area
0.25	0.9375	0.234 375
0.25	1.7500	0.437 500
0.25	2.4375	0.609 375
0.25	3.0000	0.750 000
0.25	3.4375	0.859 375
0.25	3.7500	0.937 500
0.25	3.9375	0.984 375
0.25	4.0000	1.000 000
0.25	4.0000	1.000 000
0.25	3.9375	0.984 375
0.25	3.7500	0.937 500
0.25	3.4375	0.859 375
0.25	3.0000	0.750 000
0.25	2.4375	0.609 375
0.25	1.7500	0.437 500
0.25	0.9375	0.234 375
Total Area		11.625 000

Therefore, the area is approximately 10.625 units^2. As you can see, the area is more accurate when using more partitions.

Use your calculator and do at least one of the following two questions.

1. The graph of $y = \frac{1}{2}x$ for $0 \le x \le 5$ is given as follows:

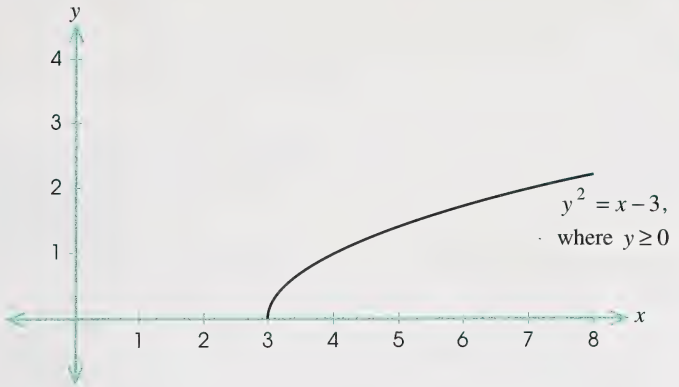


a. Calculate the sum of the areas of five inscribed rectangles and the sum of the areas of five circumscribed rectangles for the region defined by $y = \frac{1}{2}x$, $x = 0$, $x = 5$, and $y = 0$.

b. If the domain remains unchanged, use ten rectangles to find closer approximations of the areas calculated in question 1.a.

c. What is the true area?

2. The graph of $y^2 - x + 3 = 0$, where $y \geq 0$, is given as follows:



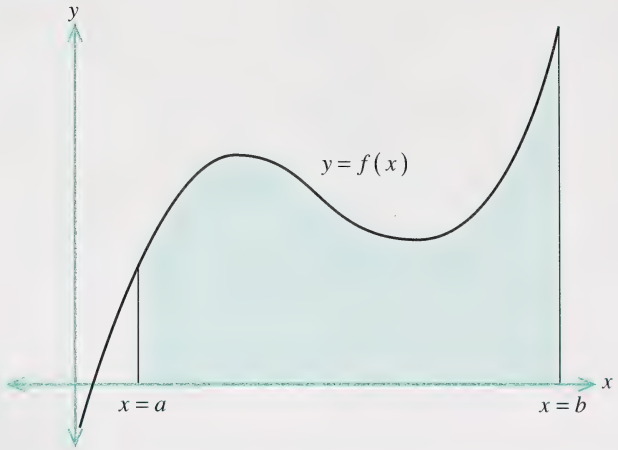
a. Calculate the sum of the areas of five inscribed rectangles and the sum of the areas of five circumscribed rectangles between $y^2 = x - 3$, $y = 0$, and $x = 8$.

b. If the domain remains unchanged, use ten rectangles to find closer approximations of the areas calculated in question 2.a.

c. What is the average area when using ten rectangles?

Check your answers by turning to the Appendix.

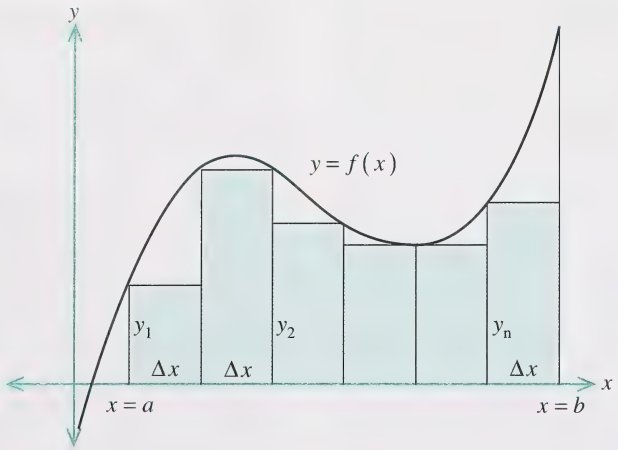
As you saw in Example 1, the greater number of inscribed or circumscribed rectangles that are used, the better the approximation is for the area under a curve.



Suppose $y = f(x)$ is a positive-valued, continuous function for $a \leq x \leq b$. Your problem is to find the area bounded above by the curve itself, below by the x-axis, on the left by $x = a$, and on the right by $x = b$.

Begin by dividing the area into n strips, and approximate the area of each strip by an inscribed rectangle of width Δx.

$$\Delta x = \frac{b - a}{n}$$



Area of the first rectangle = (length of strip)(width of strip)

$$= y_1 \Delta x$$

If c_1 is the x-value for which $f(c_1) = y_1$, then the area of the first rectangle would be $f(c_1)\Delta x$. In the diagram, it looks as though $c_1 = a$; however, it could be any x-value along the base of the first strip which would give you an inscribed rectangle.

Area of the second rectangle $= y_2 \Delta x$

$$= f(c_2)\Delta x$$

Now, c_2 is the x-value for which $f(c_2) = y_2$, the length of the second rectangle. In the preceding diagram, c_2 would be at the right edge of the strip; however, once again, it could be any x-value along the base of the second strip.

Area of the third rectangle $= y_3 \Delta x$

$$= f(c_3)\Delta x$$

$\therefore$ Area of the nth rectangle $= y_n \Delta x$

$$= f(c_n)\Delta x$$

The exact area A is approximated by the sum of these n rectangles.

$$A \doteq f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + \ldots + f(c_n)\Delta x$$

$$\doteq \sum_{i=1}^{n} f(c_i) \bullet \Delta x$$

Use the sigma notation described in Mathematics 30.

Since you are using inscribed rectangles, this approximation will be less than area A. However, you could set the same scene using circumscribed rectangles, in which case area A would be less than this approximation.

Of course, if you wanted an exact area, you would have to use more and more strips. As the number of strips increases, the width of each strip decreases.

Eventually, the area is exactly as follows:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \bullet \Delta x$$

or

$$A = \lim_{\Delta x\to 0} \sum_{a}^{b} f(c_i) \bullet \Delta x$$

If you had used circumscribed rectangles instead of inscribed rectangles, the limit would still yield the exact area A. The reason is that the function $y = f(x)$ is continuous over the interval $[a, b]$. If you pursue mathematics, you will prove this as a theorem in more advanced courses.

Try this approach in Example 2.

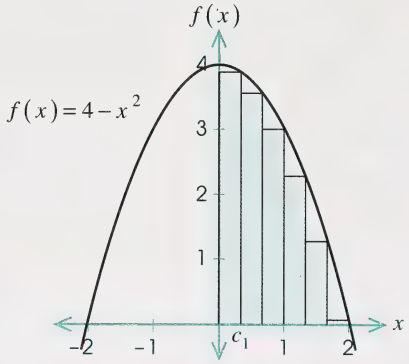
Example 2

Find the exact value of the area bounded by $f(x) = 4 - x^2$ and the x-axis. Determine that area by finding the limit of the sums of the areas of inscribed rectangles as their numbers increase without bound.

Solution

This is the same function as in Example 1. This time you will be expected to find the exact area.

Because the function is symmetric about the x-axis, simply find the area between $x = 0$ and $x = 2$ and double the result.



Divide the interval $[0, 2]$ into n strips. Because that interval is 2 units wide, each of the strips will be of width $\Delta x = \frac{2}{n}$. The x-values, c_1, c_2, c_3, ..., c_n, for which the lengths $f(c_1)$, $f(c_2)$, $f(c_3)$, ..., $f(c_n)$ of the inscribed rectangles occur, can be expressed in terms of Δx.

$c_1 = 0 + \Delta x$
$= \Delta x$

$c_2 = c_1 + \Delta x$
$= \Delta x + \Delta x$
$= 2\Delta x$

$c_3 = c_2 + \Delta x$
$= 2\Delta x + \Delta x$
$= 3\Delta x$

Therefore, $c_n = n\Delta x$.

The areas of the inscribed rectangles are as follows:

$$f(c_1)\Delta x = (4 - c_1{}^2)\Delta x \qquad f(c_2)\Delta x = (4 - c_2{}^2)\Delta x \qquad f(c_3)\Delta x = (4 - c_3{}^2)\Delta x$$
$$= [4 - (\Delta x)^2]\Delta x \qquad = [4 - (2\Delta x)^2]\Delta x \qquad = [4 - (3\Delta x)^2]\Delta x$$

$$\therefore f(c_n)\Delta x = (4 - c_n{}^2)\Delta x$$
$$= [4 - (n\Delta x)^2]\Delta x$$

Next, find the sum of the n rectangles.

$$\text{Sum} = [4 - (\Delta x)^2]\Delta x + [4 - (2\Delta x)^2]\Delta x + [4 - (3\Delta x)^2]\Delta x + \ldots + [4 - (n\Delta x)^2]\Delta x$$
$$= [4\Delta x - 1^2(\Delta x)^3] + [4\Delta x - 2^2(\Delta x)^3] + [4\Delta x - 3^2(\Delta x)^3] + \ldots + [4\Delta x - n^2(\Delta x)^3]$$
$$= \underbrace{(4\Delta x + 4\Delta x + 4\Delta x + \ldots + 4\Delta x)}_{n \text{ terms}} - [1^2 + 2^2 + 3^2 + \ldots + n^2](\Delta x)^3$$
$$= 4n\Delta x - [1^2 + 2^2 + 3^2 + \ldots + n^2](\Delta x)^3$$

Now, replace Δx by $\frac{2}{n}$. Also, the formula for the sum of the n squares is $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$. To satisfy yourself that this is the formula for the sum of the squares of the first n natural numbers, verify it for $n = 1$, $n = 2$, $n = 3$, $n = 4$, and so on.

For instance, when $n = 4$,

$$1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16$$
$$= 30$$

$$\frac{n(n+1)(2n+1)}{6} = \frac{4(4+1)(8+1)}{6}$$
$$= \frac{(4)(5)(9)}{6}$$
$$= 30$$

Return to the original problem and make the suggested substitutions.

$$\text{Sum} = 4n\frac{2}{n} - \frac{n(n+1)(2n+1)}{6} \bullet \frac{\left(2^3\right)}{n^3}$$
$$= 8 - \frac{4(n+1)(2n+1)}{3n^2}$$
$$= 8 - \frac{4\left(2n^2 + 3n + 1\right)}{3n^2}$$
$$= 8 - \frac{8}{3} - \frac{4}{n} - \frac{4}{3n^2}$$
$$= \frac{16}{3} - \frac{4}{n} - \frac{4}{3n^2}$$

Now, the exact area will be the limit of this sum as the number of strips increases without bound.

$$A = \lim_{n \to \infty} \quad \text{(sum of } n \text{ rectangles)}$$

or

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(C_i\right) \bullet \Delta x$$
$$= \lim_{n \to \infty} \left(\frac{16}{3} - \frac{1}{n} - \frac{4}{3n^2} \right)$$
$$= \frac{16}{3} - 0 - 0$$
$$= \frac{16}{3}$$

Remember, this is exactly half of the required answer.

Therefore, the area from $x = -2$ to $x = 2$ is $\frac{32}{3}$ or approximately 10.667. The approximations in Example 1 were close. Recall, for sixteen rectangles, the approximation was 10.625!

It is your turn to try this approach.

3. In Mathematics 30, the formula for the sum of the first n natural numbers is given as $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$. Using substitution, show that this result is correct for $n = 1$, $n = 2$, $n = 3$, $n = 4$, and $n = 5$.

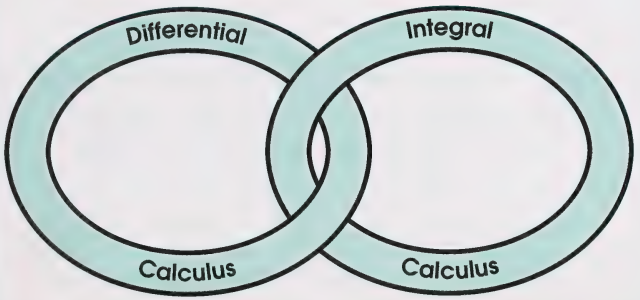
4. Sketch the area bounded above by $y = x$, on the left by $x = 1$, on the right by $x = 4$, and below by the x-axis. Determine the exact area by considering the sums of inscribed rectangles as their numbers increase without bound. In your solution, use the formula given in question 3.

Check your answers by turning to the Appendix.

You may wish to look up the method of exhaustion in a book on the history of mathematics and trace the method's evolution to techniques that are used in calculus today.

Activity 2: The Fundamental Theorem of Calculus

What Is the Link?

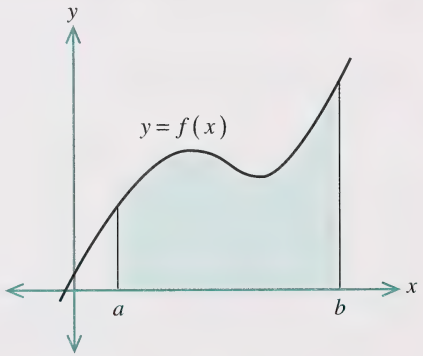


Is the problem of finding areas related to the antiderivative? Is the problem of slope somehow linked to the problem of area? What is the relationship between differential and integral calculus?

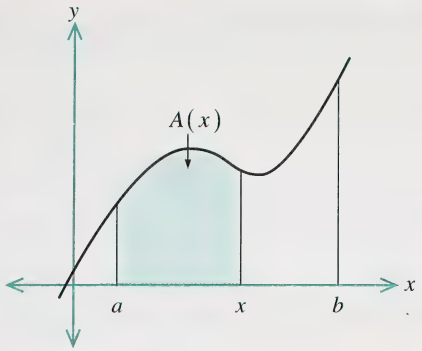
It is the consideration of these questions that links the branches of differential and integral calculus and leads to the **Fundamental Theorem of Calculus**.

The following discussion presents an argument relating derivatives and area.

Suppose that $y = f(x)$ is **continuous** and **non-negative** on the interval $a \le x \le b$. As shown in the following diagram, this defines a region between the curve and the x-axis that has a particular area.

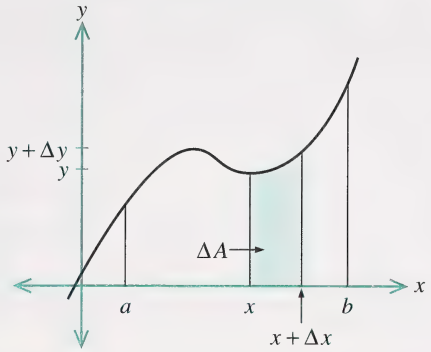


For any value x between a and b, the area under the graph from a to x is dependent on the value of x. This means that the area is some function of x. Call the function $A(x)$.



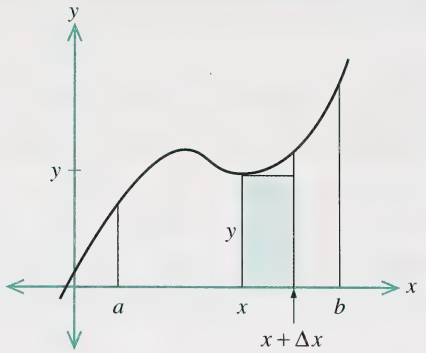
Area from a to $x = A(x)$.

Now, if you increase the value of x by Δx, the value of y will increase by some value Δy, and the area will increase by some value ΔA.



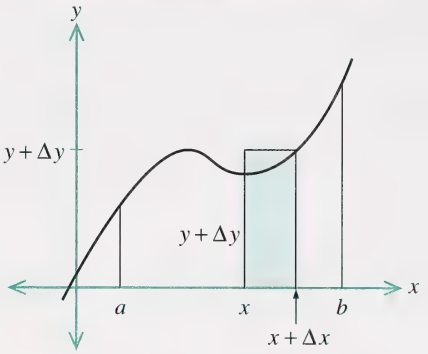
Using Δx, you can construct an inscribed rectangle and a circumscribed rectangle, such that ΔA is greater than the area of the inscribed rectangle and less than that of the circumscribed rectangle.

Inscribed Rectangles



Area of inscribed rectangle is $(y)\Delta x$.

Circumscribed Rectangles



Area of circumscribed rectangle is $(y + \Delta y)\Delta x$.

You now know that the area of an inscribed rectangle is less than or equal to ΔA. Also, the area of a circumscribed rectangle is greater than or equal to ΔA.

$$y(\Delta x) \leq \Delta A \leq (y + \Delta y)\Delta x$$

For $\Delta x \neq 0$, you can divide by Δx to get $y \leq \frac{\Delta A}{\Delta x} \leq y + \Delta y$.

Now, let $\Delta x \to 0$.

As $\Delta x \to 0$, $\Delta y \to 0$.

Therefore, $y + \Delta y \to y$.

By definition, $\lim\limits_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx}$; therefore, the inequality statement becomes $y \leq \frac{dA}{dx} \leq y$, which means $\frac{dA}{dx} = y$.

The result of the preceding discussion has important implications for you. The fact $y = \frac{dA}{dx}$ shows you that the function that defines the curve is the derivative of the area function.

To distinguish among different areas, the following symbols will be used.

- ${}_aA_b$ = area below $y = f(x)$ from $x = a$ to $x = b$
- ${}_aA_x$ = area function itself
  That is, ${}_aA_x$ = the area below $y = f(x)$ from $x = a$ to $x = x$.

Now, the area function satisfies the differential equation $\frac{d}{dx}({}_aA_x) = f(x)$, which has the initial condition ${}_aA_x = 0$ when $x = a$.

Solve the differential equation by finding the antiderivative or integral.

$$\begin{aligned} {}_aA_x &= \int f(x)\,dx \\ &= F(x) + C \end{aligned}$$

Since ${}_aA_x = 0$ when $x = a$,

$$0 = {}_aA_a \text{ and } 0 = F(a) + C$$
$$C = -F(a)$$

$$\therefore\ {}_aA_x = F(x) - F(a)$$

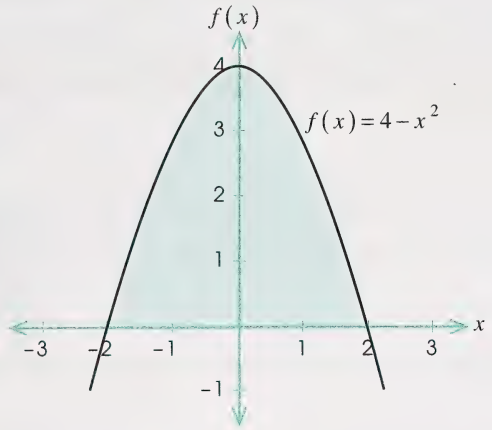
Lastly, when $x = b$, substituting into the previous equation gives ${}_aA_b = F(b) - F(a)$.

This equation states that for a positive-valued, continuous function $y = f(x)$ on the interval $[a, b]$, to find the area below that curve, to the right of $x = a$, to the left of $x = b$, and above the x-axis, find the antiderivative $F(x) + C = \int f(x)$; then calculate $F(b) - F(a)$. **Note:** a is the **lower bound** and b is the **upper bound**.

Next, you will use this approach to find the area of the function in Example 2 of Section 2: Activity 1.

Example 1

Find the area between $f(x) = 4 - x^2$ and the x-axis.



Solution

The area extends from $x = -2$ and $x = 2$.

First, determine the integral of $f(x)$.

$$\int f(x)\,dx = F(x) + C$$

$$\int \left(4 - x^2\right) dx = 4x - \frac{x^3}{3} + C$$

Now, use the integral to determine the area.

$$\begin{aligned} {}_{-2}A_2 &= F(2) - F(-2) \\ &= \left[4(2) - \frac{2^3}{3} + C\right] - \left[4(-2) - \frac{(-2)^3}{3} + C\right] \\ &= \left[8 - \frac{8}{3} + C\right] - \left[-8 + \frac{8}{3} + C\right] \\ &= 8 - \frac{8}{3} + C + 8 - \frac{8}{3} - C \\ &= 16 - \frac{16}{3} \\ &= \frac{32}{3} \end{aligned}$$

Notice that C cancels out. Since this will always occur, in future calculations it will be omitted.

This is the same answer as when the limit of the sums of the inscribed rectangles for this region was calculated. When the integral is used to evaluate a sum of this type, the following symbol may be used.

$$\int_a^b f(x)\,dx$$

The symbol $\int$ is an old-fashioned letter *S*; it originated because the integral evaluates sums.

This is a **definite integral**. It is read as **the definite integral from $x = a$ to $x = b$**. It is used to represent the area bounded by a positive and continuous function $y = f(x)$, the line $x = a$ on the left, the line $x = b$ on the right, and the x-axis below.

$$ {}_aA_b = \int_a^b f(x)\,dx = F(b) - F(a) $$

Recall that this area may be represented by a sum.

$$ {}_aA_b = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} f(c_i)\Delta x $$

Therefore, the limit of the sum may be evaluated by integration.

$$ \begin{aligned} {}_aA_b &= \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} f(c_i)\Delta x \\ &= \int_a^b f(x)\,dx \\ &= F(b) - F(a) \end{aligned} $$

This is the essence of the Fundamental Theorem of Calculus. In advanced courses you will investigate this theorem in more detail. This theorem links the process for finding sums, first posed by Eudoxus, and the problem from differential calculus of finding a function from its derivative.

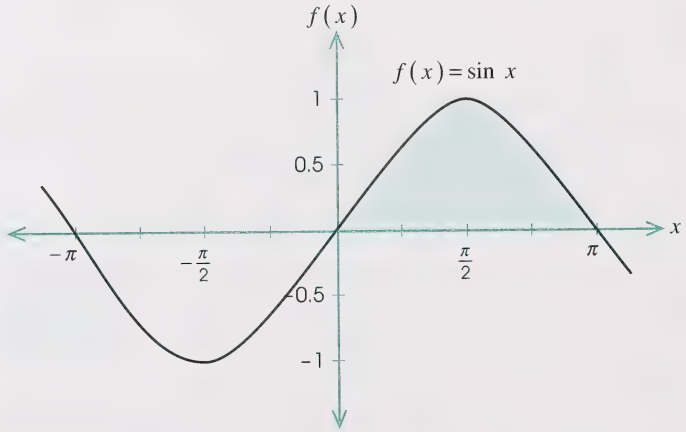
The discussion so far has been restricted to positive-valued functions. In a more formal discussion, you would see that the Fundamental Theorem of Calculus is less restrictive. Later in this module you will look at further conditions for which a function can be integrated, and how to find areas if the function crosses or lies below the x-axis.

In the next example, the definite integral is used to evaluate area.

Example 2

Find the area between the graph of $f(x) = \sin x$ and the x-axis, from $x = 0$ to $x = \pi$. Sketch the bound area.

Solution



The area may be obtained by evaluating a definite integral.

$${}_aA_b = \int_a^b f(x)\,dx = F(b) - F(a)$$

$$\begin{aligned} {}_0A_\pi &= \int_0^\pi \sin x\,dx \\ &= [-\cos x]_0^\pi \\ &= -\cos \pi - (-\cos 0) \\ &= -(-1) - (-1) \\ &= 2 \end{aligned}$$

This notation reminds you to substitute $x = \pi$ and $x = 0$ into the integral.

1. For each of the following, sketch the area bound by the function $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.

a. $f(x) = \cos 2x$, from $x = -\frac{\pi}{8}$ to $x = \frac{\pi}{4}$

b. $f(x) = \sqrt{2x-1}$, from $x = 1$ to $x = 5$

c. $f(x) = x^2$, from $x = 1$ to $x = 3$

d. $f(x) = x^3$, from $x = 0$ to $x = 2$

2. Evaluate the areas in question 1 using definite integrals.

Check your answers by turning to the Appendix.

You should now be able to describe the link between integral and differential calculus.

Activity 3: The Definite Integral and Its Properties

In the previous activity, the Fundamental Theorem of Calculus explained how to evaluate the sum of the following:

$$\lim_{\Delta x \to 0} \sum_{x=a}^{x=b} f(c_i)\Delta x$$

That sum is called the definite integral of function f from a to b. The definite integral is represented by the symbol $\int_a^b f(x)\,dx$.

To evaluate this integral, find the antiderivative F of function f, and the values of $F(b)$ and $F(a)$. Remember, a and b are the lower bound (limit) and the upper bound (limit) respectively. Then, subtract $F(a)$ from $F(b)$.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

If $y = f(x)$ is a non-negative and continuous function on the interval $[a, b]$, one interpretation of the definite integral is the area bounded by the graph of the function, the x-axis, and the vertical lines $x = a$ and $x = b$.

Because the definite integral is based on the indefinite integral F, the properties of the indefinite integral can be extended to the definite integral. In the following statements, f and g are continuous functions defined on the interval $[a, b]$; c is a real constant.

- $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
- $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

The first two properties follow directly from the properties of the indefinite integral. The third property can be verified as follows:

LS	RS
$\int_a^b f(x)\,dx$	$-\int_b^a f(x)\,dx$
$= F(b) - F(a)$	$= -[F(a) - F(b)]$
	$= -F(a) + F(b)$
	$= F(b) - F(a)$

LS = RS

This third property says that switching the limits of integration changes the sign of the integral.

In the examples and questions that follow, you will apply these properties.

Example 1

Verify that $\int_1^2 3x\,dx = 3\int_1^2 x\,dx$.

Solution

LS	RS
$\int_1^2 3x\,dx$	$3\int_1^2 x\,dx$
$= \left[\frac{3x^2}{2}\right]_1^2$	$= 3\left[\frac{x^2}{2}\right]_1^2$
$= \frac{3(2)^2}{2} - \frac{3(1)^2}{2}$	$= 3\left[\frac{2^2}{2} - \frac{1^2}{2}\right]$
$= \frac{12}{2} - \frac{3}{2}$	$= 3\left[\frac{4}{2} - \frac{1}{2}\right]$
$= \frac{9}{2}$	$= 3\left(\frac{3}{2}\right)$
	$= \frac{9}{2}$

LS = RS

Now it's your turn.

1. Verify that $\int_1^2 \left(2x + 3x^2\right)\, dx = \int_1^2 2x\, dx + \int_1^2 3x^2\, dx$.

2. Show that $\int_{-1}^{2} (3 + 2x)\, dx = -\int_{2}^{-1} (3 + 2x)\, dx$.

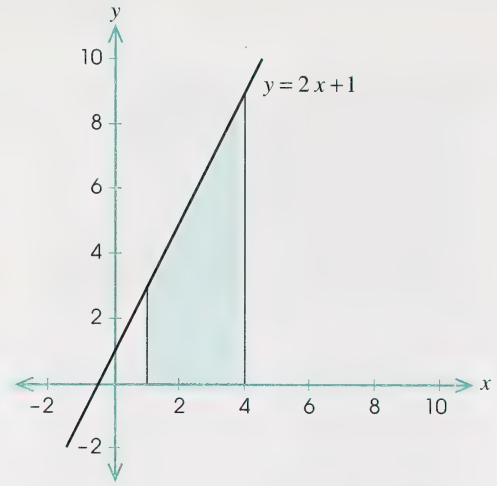
Check your answers by turning to the Appendix.

You may use a graphing calculator to check your answers to questions 1 and 2. The procedure for using the CASIO *fx-7700G* graphing calculator is given in Example 2. If you have a different type of graphing calculator, consult the owner's manual.

Example 2

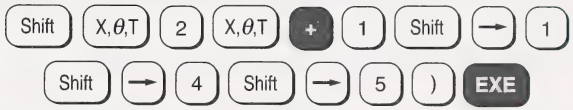
Sketch the area bounded by $y = 2x + 1$, $x = 1$, $x = 4$, and the x-axis. Using a graphing calculator, find an approximation for that area. Compare your answer to the exact area obtained from the definite integral.

Solution



To obtain an approximation on the CASIO *fx-7700G* graphing calculator, enter the sequence $\int dx(2x+1,\ 1,\ 4,\ n)$. In this sequence, $f(x)$ is the given function, a is the lower bound, b is the upper bound, and n is the number of strips into which the area is divided. The number of strips may be any natural number from 1 to 9 inclusive. In this example, $f(x) = 2x + 1$, $a = 1$, and $b = 4$. Try $n = 5$.

In this example, the key sequence is as follows:



The answer is 18, which happens to be the exact answer in this instance.

$$
\begin{aligned}
{}_1A_4 &= \int_1^4 (2x+1)\,dx \\
&= \left[x^2 + x\right]_1^4 \\
&= \left(4^2 + 4\right) - \left(1^2 + 1\right) \\
&= 20 - 2 \\
&= 18
\end{aligned}
$$

Example 3

Evaluate $\int_{-1}^{1} \left[\frac{x}{\left(x^2+4\right)^2} \right] dx$.

Solution

$$\int \left[\frac{x}{\left(x^2+4\right)^2} \right] dx = \int x\left(x^2+4\right)^{-2}\,dx$$

Use comparison techniques to find the antiderivative of $y = x\left(x^2+4\right)^{-2}$. Assume it is of the form $F(x) = \frac{a\left(x^2+4\right)^{-1}}{-1}$.

$$
\begin{aligned}
\therefore \frac{d}{dx}F(x) &= -a(-1)\left(x^2-4\right)^{-2}(2x) \\
&= 2ax\left(x^2-4\right)^{-2}
\end{aligned}
$$

Equate this expression to the integrand.

$$
\begin{aligned}
2ax\left(x^2-4\right)^{-2} &= x\left(x^2+4\right)^{-2} \\
2a &= 1 \\
a &= \frac{1}{2}
\end{aligned}
$$

$$\therefore F(x) = \frac{-1\left(x^2+4\right)^{-1}}{2}$$

$$
\begin{aligned}
\int_{-1}^{1} \left[\frac{x}{\left(x^2+4\right)^2} \right] dx &= \left[-\frac{1}{2}\left(x^2+4\right)^{-1} \right]_{-1}^{1} \\
&= -\frac{1}{2}\left(1^2+4\right)^{-1} - \left\{ -\frac{1}{2}\left[(-1)^2+4\right]^{-1} \right\} \\
&= -\frac{1}{10} + \frac{1}{10} \\
&= 0
\end{aligned}
$$

Before going further, view the video titled *Integration* from the *Catch 31* series, ACCESS Network. This video provides an excellent summary of the discussion of the first three activities of this section. You may wish to make notes or replay segments so that you will master the content. This video is available from the Learning Resources Distributing Centre.

Use a graphing calculator to check your answers to question 3.

3. Determine the following:

a. $\int_{-3}^{2}\left(3x^2-4\right)\,dx$

b. $\int_{-1}^{3}\left[\frac{1}{(3x-2)^2}\right]dx$

c. $\int_{3}^{4}\left(2x^2+3\right)^2\,dx$

d. $\int_{7}^{12}(x-3)^{\frac{1}{2}}\,dx$

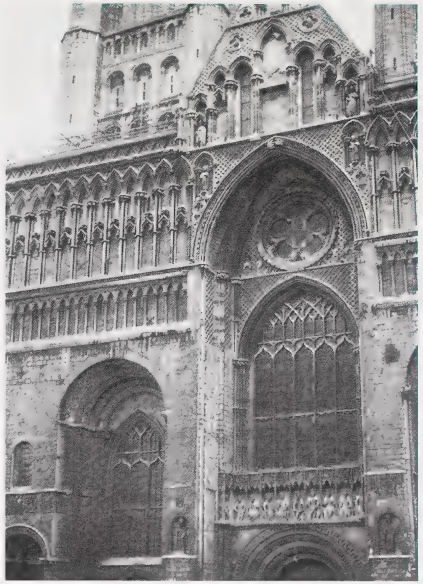
e. $\int_{-2}^{1}2x\left(x^2-1\right)^{\frac{1}{2}}\,dx$

f. $\int_{-1}^{2}(5-x)^{\frac{1}{2}}\,dx$

Check your answers by turning to the Appendix.

Review the properties of the definite integral. Can you describe these properties using your own words?

Activity 4: Areas Bounded by the *x*-axis



Architects often design buildings with windows of irregular shapes. How would the architect determine the area of the window in the photograph if the top of that window were parabolic? Could calculus be used?

You have discovered in previous activities that for a positive-valued, continuous function $y = f(x)$ defined on the interval $[a, b]$, the area bounded above by the curve, below by the x-axis, on the left by $x = a$, and on the right by $x = b$, is given by the definite integral from a to b.

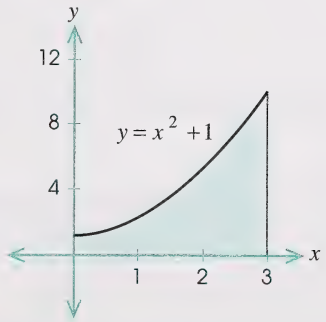
$$_aA_b = \int_a^b f(x)\,dx = F(b) - F(a)$$

Example 1

Determine the area bounded by $y = x^2 + 1$, the x-axis, and the lines $x = 0$ and $x = 3$.

Solution

The curve $y = x^2 + 1$ is continuous and positive on the interval $0 \le x \le 3$ as shown in the diagram.



$$_0A_3 = \int_0^3 \left(x^2 + 1\right) dx = \left[\frac{x^3}{3} + x\right]_0^3$$
$$= (9 + 3) - (0 + 0)$$
$$= 12$$

Therefore, the area is 12.

Is the area always equal to the definite integral?

If you think the area is equal to the definite integral only when the function is continuous and positive, you are correct.

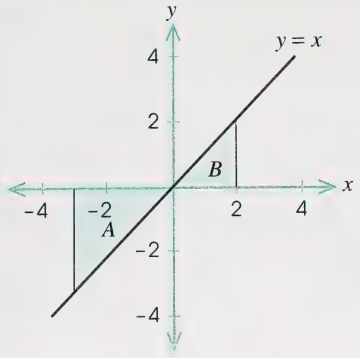
What about regions below the x-axis on the interval or if the graph crosses the axis on the interval?

Investigate an example to see if you can determine the area in these cases.

Example 2

Determine the area enclosed between the x-axis and the line defined by $y = x$ on the interval $-3 \le x \le 2$.

Solution



Using the formula for the area of a triangle, the area of region A is 4.5 and the area of region B is 2. The total area is 6.5.

If you evaluate $\int_{-3}^{2} x\,dx = \left[\frac{x^2}{2}\right]_{-3}^{2}$, you get a value of -2.5.

What happened? How can you fix it?

Look at the interval where the region is below the x-axis.

$$\int_{-3}^{0} x\,dx = \left[\frac{x^2}{2}\right]_{-3}^{0}$$
$$= \frac{0^2}{2} - \frac{(-3)^2}{2}$$
$$= -4.5$$

This is where you will find the problem. Regions below the x-axis correspond to negative evaluations for the definite integral. You can fix this by using absolute values.

$$_{-3}A_0 = \left|\int_{-3}^{0} x\,dx\right| \qquad _0A_2 = \left|\int_0^2 x\,dx\right|$$
$$= \left|\frac{x^2}{2}\right|_{-3}^{0} \qquad = \left|\frac{x^2}{2}\right|_0^2$$
$$= |-4.5| \qquad = 2$$
$$= 4.5$$

Therefore, the total area is $_{-3}A_0 + {}_0A_2 = 4.5 + 2 = 6.5$.

It is critical to know where the curve is above the x-axis and where it is below the x-axis on a given interval. A sketch of the function is helpful when determining this.

Example 3

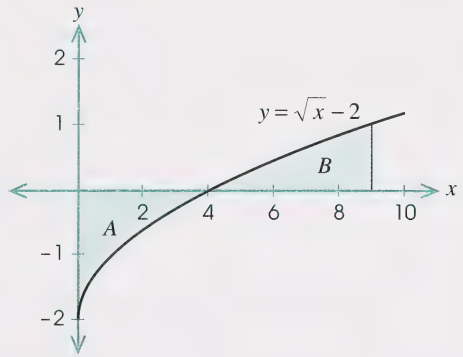
Find the area bounded by $y = \sqrt{x} - 2$ and the x-axis over the interval $0 \le x \le 9$.

Solution

First, sketch the curve, noting particularly any x-intercept in the interval $0 \le x \le 9$. Shade the region where the area is to be determined.

To find the x-intercept, let $y = 0$.

$$y = \sqrt{x} - 2$$
$$0 = \sqrt{x} - 2$$
$$\sqrt{x} = 2$$
$$x = 4$$



Region A is below the x-axis.

$$
\begin{aligned}
{}_0A_4 &= \left| \int_0^4 \left(x^{\frac{1}{2}} - 2 \right) dx \right| \\
&= \left| \frac{2}{3}x^{\frac{3}{2}} - 2x \right|_0^4 \\
&= \left| \left[\frac{2}{3}(4)^{\frac{3}{2}} - 2(4) \right] - \left[\frac{2}{3}(0)^{\frac{3}{2}} - 2(0) \right] \right| \\
&= \left| \frac{16}{3} - 8 \right| \\
&= \left| -\frac{8}{3} \right| \\
&= \frac{8}{3}
\end{aligned}
$$

Region B is above the x-axis.

$$
\begin{aligned}
{}_4A_9 &= \int_4^9 \left(x^{\frac{1}{2}} - 2 \right) dx \\
&= \left[\frac{2}{3}x^{\frac{3}{2}} - 2x \right]_4^9 \\
&= \left[(18 - 18) - \left(\frac{16}{3} - 8 \right) \right] \\
&= \left(0 + \frac{8}{3} \right) \\
&= \frac{8}{3}
\end{aligned}
$$

$$
\begin{aligned}
\text{Total Area} &= {}_0A_4 + {}_4A_9 \\
&= \frac{8}{3} + \frac{8}{3} \\
&= \frac{16}{3}
\end{aligned}
$$

Therefore, the total area is $\frac{16}{3}$.

No matter how complex the function, you can follow the procedure shown in Example 3. Consider the next example.

Example 4

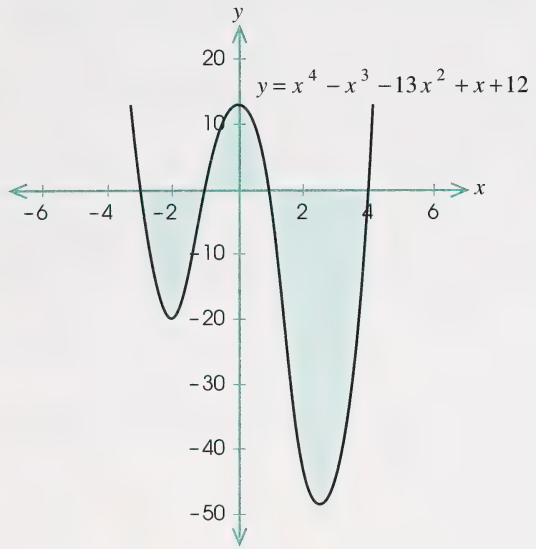
Find the area of the regions bounded by the x-axis and the curve defined by $y = x^4 - x^3 - 13x^2 + x + 12$.

Solution

Using the Factor Theorem and synthetic division, you can determine the roots of the function. The factors of the polynomial follow:

$$\begin{aligned} y &= x^4 - x^3 - 13x^2 + x + 12 \\ &= (x-1)(x+1)(x+3)(x-4) \end{aligned}$$

As indicated by the factors, the graph crosses the x-axis at $x=-3$, $x=-1$, $x=1$, and $x=4$. Using your knowledge about polynomial functions, sketch the graph.



In the graph, two regions are below the x-axis and one region is above the x-axis.

$$\begin{aligned} {}_{-3}A_{-1} &= \left| \int_{-3}^{-1} \left(x^4 - x^3 - 13x^2 + x + 12 \right) dx \right| \\ &= \left| \frac{x^5}{5} - \frac{x^4}{4} - \frac{13x^3}{3} + \frac{x^2}{2} + 12x \right|_{-3}^{-1} \\ &= \left| -24.2\overline{6} \right| \\ &= 24.2\overline{6} \end{aligned}$$

$$\begin{aligned} {}_{-1}A_{1} &= \int_{-1}^{1} \left(x^4 - x^3 - 13x^2 + x + 12 \right) dx \\ &= \left[\frac{x^5}{5} - \frac{x^4}{4} - \frac{13x^3}{3} + \frac{x^2}{2} + 12x \right]_{-1}^{1} \\ &= 15.7\overline{3} \end{aligned}$$

$$\begin{aligned} {}_{1}A_{4} &= \left| \int_{1}^{4} \left(x^4 - x^3 - 13x^2 + x + 12 \right) dx \right| \\ &= \left| \frac{x^5}{5} - \frac{x^4}{4} - \frac{13x^3}{3} + \frac{x^2}{2} + 12x \right|_{1}^{4} \\ &= 88.65 \end{aligned}$$

$$\begin{aligned} \text{Total Area} &= {}_{-3}A_{-1} + {}_{-1}A_{1} + {}_{1}A_{4} \\ &= 24.2\overline{6} + 15.7\overline{3} + 88.65 \\ &\doteq 128.65 \end{aligned}$$

Therefore, the total area is about 128.65.

Now try the following questions.

1. Determine the area enclosed by the x-axis, the graph of $y = f(x)$, and the two vertical lines given.

a. $f(x) = x - 3,\ x = 2,\ x = 4$

b. $f(x) = x^3 - 4x^2,\ x = -3,\ x = 3$

c. $f(x) = \dfrac{3x}{\sqrt{x^2 + 2}},\ x = 0,\ x = 3$

d. $f(x) = \sin \dfrac{x}{2},\ x = 0,\ x = 2\pi$

2. Determine the area enclosed by the x-axis and the graph of y defined as follows.

a. $y = -2x^2 + x + 3$ **b.** $y = 3x^2 - x^3$

3. Determine the area enclosed by the x-axis, the graph of $y = f(x)$, and the two vertical lines given.

a. $y = -2x^2 - 3x + 2,\ x = -2,\ x = 1$

b. $y = \sqrt{x - 1},\ x = 2,\ x = 5$

c. $y = \sec^2 x,\ x = -\dfrac{\pi}{4},\ x = \dfrac{\pi}{4}$

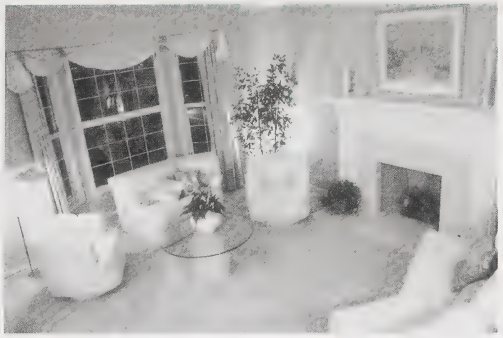
4. Determine the area enclosed by the x-axis and the graph of y defined as follows.

a. $y = x^4 - 9x^2$ **b.** $y = (x - 2)^2 - 3$

Check your answers by turning to the Appendix.

In this activity you explored techniques for finding areas between a curve and the x-axis. How could these techniques be applied to architectural design?

Activity 5: Areas Between Curves



Whether you are painting or wallpapering a room, laying concrete for a curved sidewalk or driveway, cutting out clothing patterns from fabric, or performing any other task for which knowing the surface area is important, what are the techniques for determining those areas?

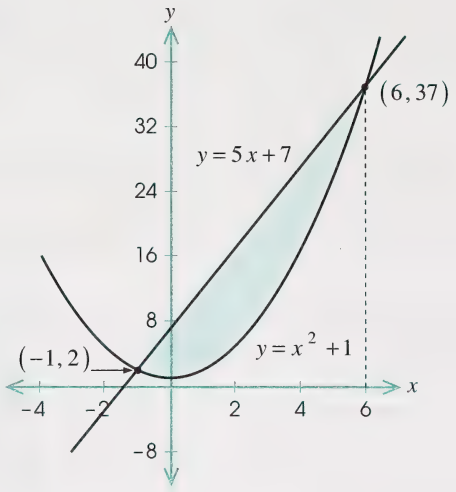
In this activity you will investigate calculus techniques for finding the area bounded by two curves.

Example 1

Determine the area between the curves defined by $y = x^2 + 1$ and $y = 5x + 7$.

Solution

Sketch the curves and determine the region involved.



From the diagram, the points of intersection of the graphs of the functions are $(-1, 2)$ and $(6, 37)$.

$$\begin{cases} y = 5x + 7 \\ y = x^2 + 1 \end{cases}$$

At the points of intersection the y-values will be the same.

$$\begin{aligned} y &= y \\ 5x+7 &= x^2+1 \\ x^2-5x-6 &= 0 \\ (x-6)(x+1) &= 0 \\ x-6=0 \text{ or } x+1 &= 0 \\ x=6 \qquad x &= -1 \end{aligned}$$

The corresponding y-values are determined by substitution into either of the defining equations.

When $x=6$, $y=37$.
When $x=-1$, $y=2$.

Now, $\int_{-1}^{6}(5x+7)\,dx$ is the area bounded by the line and the x-axis over the interval $-1 \le x \le 6$, and $\int_{-1}^{6}\left(x^2+1\right)\,dx$ is the area under the parabola over that same interval.

How can you determine the area between the two curves?

If you said subtract, you are right. The area between these curves is given by the following:

$$\int_{-1}^{6}(5x+7)\,dx - \int_{-1}^{6}\left(x^2+1\right)\,dx$$

$$\begin{aligned} A &= \int_{-1}^{6}\left[(5x+7)-\left(x^2+1\right)\right]\,dx \\ &= \int_{-1}^{6}\left(-x^2+5x+6\right)\,dx \\ &= \left[\frac{-x^3}{3}+\frac{5x^2}{2}+6x\right]_{-1}^{6} \\ &= 57\frac{1}{6} \end{aligned}$$

The area is $57\frac{1}{6}$.

When you encounter something new, you should try to think about what other possibilities there are. Does the position of the region relative to the x-axis affect the area or the way you calculate it?

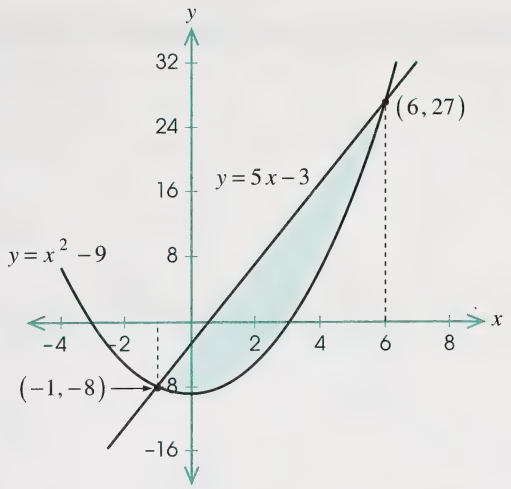
Consider the following example where the region goes below the x-axis.

Example 2

Determine the area between the curves defined by $y=x^2-9$ and $y=5x-3$.

Solution

Sketch the graphs. Determine the area involved.



How does this compare with the graph in Example 1? It is the same, except it has been shifted down 10 units.

You are dealing with a region which has been **translated** 10 units down on the coordinate plane. The area of the region is not changed by this translation; thus, it should be $57\frac{1}{6}$ as it was in Example 1.

The points of intersection occur where $y = y$.

$$x^2 - 9 = 5x - 3$$
$$x^2 - 5x - 6 = 0$$
$$(x-6)(x+1) = 0$$
$$x - 6 = 0 \quad \text{or} \quad x + 1 = 0$$
$$x = 6 \qquad x = -1$$

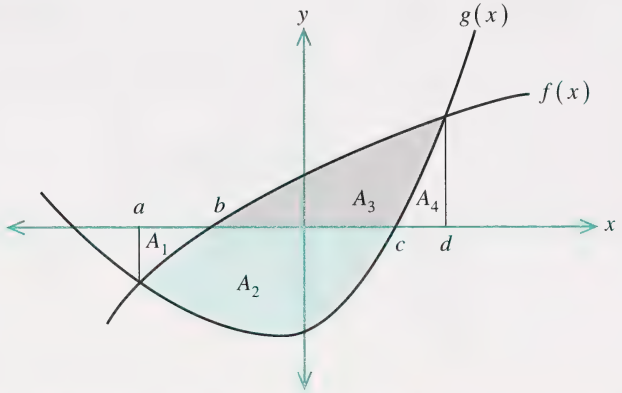
Using substitution, you get $(6, 27)$ and $(-1, -8)$ as the points of intersection.

$$\int_{-1}^{6}(5x-3)\,dx - \int_{-1}^{6}\left(x^2-9\right)\,dx = \int_{-1}^{6}\left[(5x-3)-\left(x^2-9\right)\right]\,dx$$
$$= \int_{-1}^{6}\left(-x^2+5x+6\right)\,dx$$
$$= 57\frac{1}{6}$$

The enclosed area is $57\frac{1}{6}$.

You can ignore the position of the curves relative to the x-axis when you are looking for the area between two curves.

Look at a more general case to justify this observation.

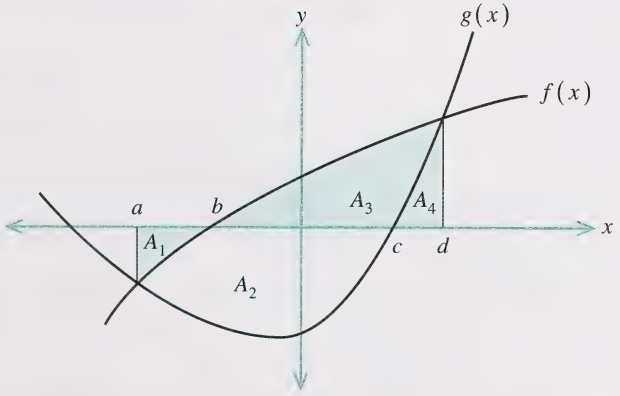


In the diagram there are two curves, defined by $y = f(x)$ and $y = g(x)$, which intersect where $x = a$ and $x = d$. The area of the region between the curves is given by $A_2 + A_3$. In the interval $a \le x \le d$, $f(b) = 0$ and $g(c) = 0$; that is, both curves cross the x-axis.

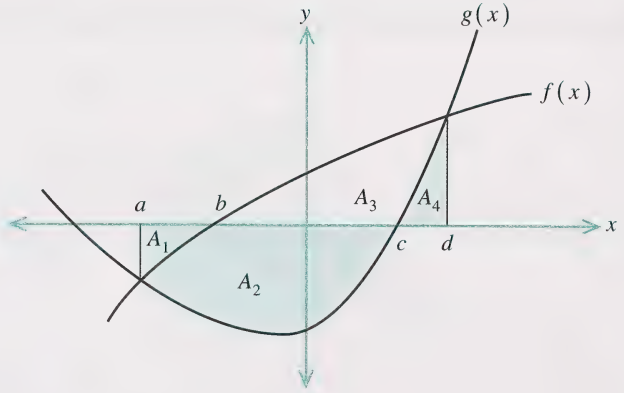
Recall the relationship between the definite integral and area.

- If $f(x) > 0$ on the interval $a \le x \le b$, then $\int_a^b f(x)\,dx = {}_aA_b$.
- If $f(x) < 0$ on the interval $a \le x \le b$, then $\int_a^b f(x)\,dx = -\,{}_aA_b$.

Use this relationship and the diagram.



$$\int_a^d f(x)\,dx = -A_1 + A_3 + A_4$$



$$\int_a^d g(x)\,dx = -A_1 - A_2 + A_4$$

You want to find the area between the curves; that is, $A_3 + A_2$.

$$\int_a^d f(x)\,dx - \int_a^d g(x)\,dx = \left(-A_1 + A_3 + A_4\right) - \left(-A_1 - A_2 + A_4\right)$$
$$= A_3 + A_2$$

In general, if the curves defined by $y = f(x)$ and $y = g(x)$ are such that $f(x) > g(x)$ in the interval $a \le x \le b$. Then the area between the curves over that interval is the integral of the top curve minus the bottom curve.

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \left[f(x) - g(x)\right] dx$$

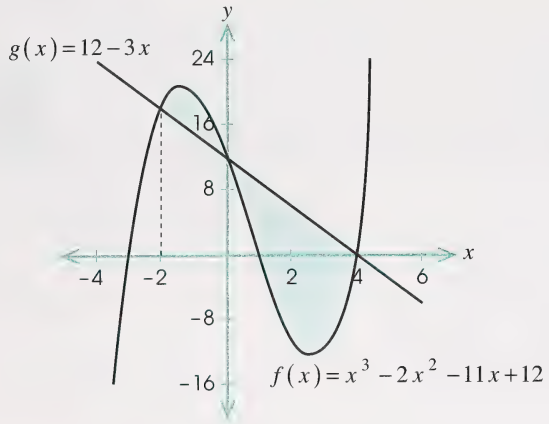
Look at one more example.

Example 3

Find the total area between the curves defined by $f(x)=x^3-2x^2-11x+12$ and $g(x)=12-3x$.

Solution

As shown in the graph, there are two regions.



The curves intersect when $f(x)=g(x)$.

$$x^3-2x^2-11x+12=12-3x$$
$$x^3-2x^2-8x=0$$
$$x(x+2)(x-4)=0$$
$$x=0 \text{ or } x+2=0 \text{ or } x-4=0$$
$$x=-2 \qquad x=4$$

The points of intersection are $(-2,18)$, $(0,12)$, and $(4,0)$.

For $-2<x<0$, $f(x)$ defines the top curve since $f(x)>g(x)$.

$$_{-2}A_0=\int_{-2}^{0}[f(x)-g(x)]\,dx$$
$$_{-2}A_0=\int_{-2}^{0}\left[\left(x^3-2x^2-11x+12\right)-(12-3x)\right]dx$$
$$=\int_{-2}^{0}\left(x^3-2x^2-8x\right)dx$$
$$=\left[\frac{x^4}{4}-\frac{2x^3}{3}-4x^2\right]_{-2}^{0}$$
$$=\frac{20}{3}$$

For $0 < x < 4$, $g(x)$ defines the top curve.

$$
\begin{aligned}
{}_0A_4 &= \int_0^4 \left[g(x) - f(x) \right] dx \\
&= \int_0^4 \left[(12 - 3x) - \left(x^3 - 2x^2 - 11x + 12 \right) \right] dx \\
&= \int_0^4 \left(-x^3 + 2x^2 + 8x \right) dx \\
&= \left[\frac{-x^4}{4} + \frac{2x^3}{3} + 4x^2 \right]_0^4 \\
&= \frac{128}{3}
\end{aligned}
$$

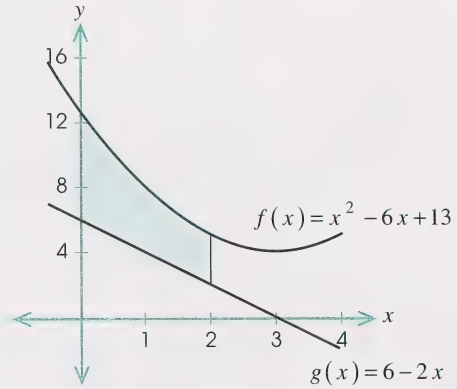
The total area is $\frac{20}{3} + \frac{128}{3} = \frac{148}{3} = 49\frac{1}{3}$.

It is important to know if the curves intersect in an interval, and if so, it is important to know which is the **top** curve and the **bottom** curve on that interval. The curves need not intersect if an interval is specified (as illustrated in the following example).

Example 4

Find the area bounded by the curve defined by $f(x) = x^2 - 6x + 13$ and the line defined by $g(x) = 6 - 2x$ over the interval $0 \le x \le 2$.

Solution



As shown in the preceding diagram, the graphs do not intersect; but the region is well-defined using the given curves and the vertical lines $x = 0$ and $x = 2$.

For $0 \le x \le 2$, $f(x) > g(x)$. The area is given by the following:

$$
\begin{aligned}
{}_0A_2 &= \int_0^2 \left[f(x) - g(x) \right] dx \\
&= \int_0^2 \left[\left(x^2 - 6x + 13 \right) - (6 - 2x) \right] dx \\
&= \int_0^2 \left(x^2 - 4x + 7 \right) dx \\
&= \left[\frac{x^3}{3} - 2x^2 + 7x \right]_0^2 \\
&= \frac{26}{3}
\end{aligned}
$$

Before doing the following questions, watch the first half of the video titled *Applications of Integrals to Area and Motion* from the *Catch 31* series, ACCESS Network. The first half of this video deals with finding the area between a curve and the x-axis, and the area between two curves. The approach used varies slightly from the approach presented in this module, giving you an alternate perspective. You may use either approach in your questions. This video is available from the Learning Resources Distributing Centre.

1. Find the area between the curves defined by the following:

a. $y = -x^2 + 4$ and $y = 2x + 1$

b. $y = x^3 - 5x^2 + 2x + 8$ and $y = -2x + 8$

2. Find the area bounded by $y = x^2 + 4x - 9$ and the line defined by $y = 2x - 1$, over the interval $-3 \le x \le 1$.

3. Find the area bounded by $y = 3x + 1$ and $y = \sqrt{x}$, over the interval $0 \le x \le 5$.

4. Find the area enclosed by the graphs of $y = x^2 - 2x + 5$ and $y = -x^2 + 4x + 1$.

5. Find the area enclosed by the following:

a. $y = \sin x$ and $y = \cos x$, between $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$

b. $y = 2\cos 2x$ and $y = -2\cos 2x$, between $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$

Check your answers by turning to the Appendix.

Being able to determine areas between curves is important in the construction trades. Accurate estimates of materials for a driveway or a curved racetrack, for example, help control costs.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

If you have access to a computer and the software packages Zap-a-Graph™ and the *Area Explorer* on the *CAI-Math 31* disk, do both Part A and Part B. If you do not have access to these software packages, then do Part A only.

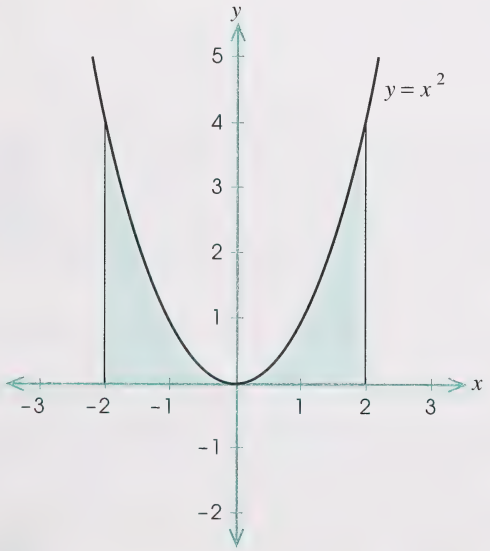
Part A

If the area you are determining by calculus is symmetric about the y-axis, or about the origin, there is a shortcut you can take that simplifies the evaluation of the definite integral.

Example

Find the area bounded by $y = x^2$, the x-axis, and the vertical lines $x = -2$ and $x = 2$.

Solution



Since the area is symmetric about the y-axis, instead of integrating from -2 to $+2$, simply evaluate the definite integral from 0 to 2 and double the result.

$$
\begin{aligned}
{}_{-2}A_2 &= 2\ {}_0A_2 \\
&= 2\int_0^2 x^2\ dx \\
&= 2\left[\frac{x^3}{3}\right]_0^2 \\
&= \frac{2}{3}\left(2^3 - 0^3\right) \\
&= \frac{16}{3}
\end{aligned}
$$

The advantage arises in the evaluation of the integral for $x = 0$. This evaluation is often simpler than an evaluation involving negative numbers or fractions.

Now try the following:

1. Find the area enclosed by the curves $y = x^2$ and $y = 4 - x^2$.

2. Find the area between $y = \sin x$ and the x-axis, from $x = -\pi$ and $x = \pi$.

Check your answers by turning to the Appendix.

Part B

One of the features of Zap-a-Graph™ and of the *Area Explorer* on the *CAI Math 31* disk is their ability to estimate the area between a curve and the x-axis.

Zap-a-Graph™ allows you to divide the area below a continuous section of a function into any number of strips. The estimated area is the sum of the areas of the inscribed rectangles approximating each strip. By choosing larger and larger numbers of strips, you can approach the exact value.

Similarly, the *Area Explorer* also allows you to divide the area between a polynomial of degree 4 or less and the x-axis into strips. Two estimates for the area are given: the smaller for inscribed rectangles, and the larger for circumscribed rectangles. The actual area lies between those two estimates. Again, more accurate estimates can be obtained by subdividing the area into a greater number of strips.

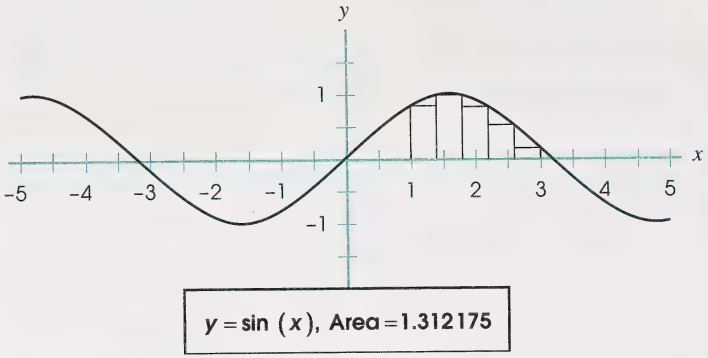
You will use Zap-a-Graph™ in the next example to estimate a definite integral.

Example

Estimate, using Zap-a-Graph™, the area between the graph of $y = 3 \sin x$, the x-axis, and the vertical lines $x = 1$ and $x = 3$.

Solution

Enter the function $y = \sin x$ on Zap-a-Graph™; pull down the Options window and select "Area". The computer then prompts you to select the lower bound (enter **1**), the upper bound (enter **3**), and the number of intervals (enter **5** as your first selection).



Your first estimate (five strips) will appear on the computer screen as shown in the preceding diagram. The following table shows the changing estimates as the number of strips increases.

Number of Strips	Estimate
5	1.312 175
10	1.423 535
100	1.517 248

The actual area, correct to nine decimal places, is 1.530 294 802.

At this point you may wish to return to Activity 1 of this section, and see how the estimates of these software packages compare to the exact values.

Enrichment

If two curves are given, the points of intersection of these two curves may not represent the left- or right-hand extremes of the region. If you want to find the area enclosed by these two curves, then you have a new type of problem.

Example

Find the area between the graphs of $y^2 = x+3$ and $y = x+1$.

Solution

The points of intersection must be determined.

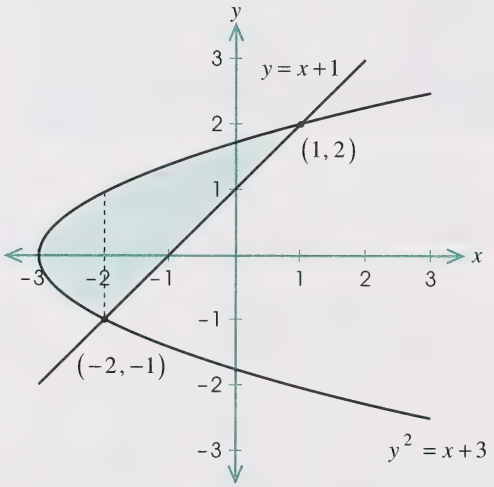
$$y^2 = x+3 \quad (1)$$
$$y = x+1 \quad (2)$$

$$y^2 = y^2$$
$$\therefore (x+1)^2 = x+3$$
$$x^2+2x+1 = x+3$$
$$(x+2)(x-1) = 0$$
$$x^2+x-2 = 0$$

$$x+2=0 \quad \text{or} \quad x-1=0$$
$$x=-2 \qquad x=1$$

At $x=-2$, $y=-1$; at $x=1$, $y=2$.

Thus, $(-2,-1)$ and $(1,2)$ are the two points of intersection.



The extremes are $(-3,0)$ and $(1,2)$. You have to separate the region over $-3 \le x \le -2$ into two parts. The top half of $y^2 = x+3$ is $y = \sqrt{x+3}$ and the bottom half of $y^2 = x+3$ is $y = -\sqrt{x+3}$; thus, the area can be determined between $y = \sqrt{x+3}$ and $y = -\sqrt{x+3}$ over the interval $-3 \le x \le -2$. Then the area can be calculated between $y = \sqrt{x+3}$ and $y = x+1$ for the interval $-2 \le x \le 1$.

$$A = \int_{-3}^{-2}\left[\sqrt{x+3} - \left(-\sqrt{x+3}\right)\right]\,dx + \int_{-2}^{1}\left[\sqrt{x+3} - (x+1)\right]\,dx$$

$$= \int_{-3}^{-2}\left(2\sqrt{x+3}\right)\,dx + \int_{-2}^{1}\left(\sqrt{x+3} - x - 1\right)\,dx$$

$$= \left[\frac{2(x+3)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_{-3}^{-2} + \left[\frac{2}{3}(x+3)^{\frac{3}{2}} - \frac{x^2}{2} - x\right]_{-2}^{1}$$

$$= \frac{4}{3}[(+1) - 0] + \frac{2}{3}(4)^{\frac{3}{2}} - \frac{1}{2} - 1 - \frac{2}{3} + \frac{(-2)^2}{2} + (-2)$$

$$= \frac{4}{3} + \frac{16}{3} - \frac{1}{2} - 1 - \frac{2}{3} + 2 - 2$$

$$= 4\frac{1}{2}$$

Try the following question.

Find the area between the graphs of $y^2 = x + 1$ and $y = x - 1$.

Check your answers by turning to the Appendix.

Conclusion

In the first activity of this section, you found that the area under a curve over a given interval can be subdivided into strips, and that each strip can be approximated by a rectangle. You also discovered that the sum of those rectangles approximates the total area. You established that the exact area will lie between two bounds: greater than or equal to the sum of inscribed rectangles and less than or equal to the sum of circumscribed rectangles. You could then use the limit of the sums of either inscribed or circumscribed rectangles to find the exact area.

The discussion of the Fundamental Theorem of Calculus linked the area limit to the antiderivative of a function describing a curve. In particular, you saw that for a non-negative function $y = f(x)$, defined on the interval $[a, b]$, the area bounded above by the function, below by the x-axis, on the left by $x = a$, and on the right by $x = b$ was the definite interval, $\int_a^b f(x)\,dx = F(b) - F(a)$.

You calculated the definite integral for polynomial, rational, and trigonometric functions. You used the following properties to simplify more complicated definite integrals.

- $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
- $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

In the last two activities, you applied the definite integral to find areas between a curve $y = f(x)$ and the x-axis if the function had either a constant sign over the given interval or if the curve crossed the x-axis in that interval. Also, you found areas between two curves either between two stated values or between their intersection points.

Practical application of the techniques in this section are occasionally limited by the difficulty in determining the equations of the curves bounding an irregular area. Farmers, familiar with their land, have no difficulty estimating the area of a grain field left to be harvested. But, could the techniques of this section be applied to calculate such an area more accurately?

If you do not feel comfortable with any of these concepts, make certain you review the relevant topics.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Section 3: Applications of Integration

Part of the exhilaration of downhill skiing is the sense of speed. What governs a skier's speed? the slope of the hill? the angle at which the skier attacks the slope? In the photograph, the skier's speed is controlled by carving out a path down the mountainside, using the relationships among force, acceleration, velocity, and distance.

In Module 6, you investigated the relationships among distance, velocity, and acceleration. You discovered that velocity and acceleration are derivatives. In the first two activities in this section you will be shown how to use antiderivatives to find velocity from acceleration, and position from velocity.

In Activity 3, you will be introduced to examples of simple harmonic motion—a mass vibrating at the end of a spring, a vibrating violin string, a pendulum. Again, you will relate position, velocity, and acceleration to solving second-order differential equations that model this motion.

Activity 4 looks at the average value of a function as an application of integration. Real-world examples, such as average temperature and average velocity, are used to illustrate procedures and the relationship between the average value and the graph of the function being averaged.

This section will prepare you for further work in integral calculus and its application to the sciences and real-world examples.

Activity 1: Position from Velocity

How far will Anna coast on her carpet when she reaches the bottom of the hill? To help answer this question, you will need to know how fast she is travelling.

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Recall from Module 6 that, for objects travelling along a line, velocity v is defined as a change in position (displacement) s over time t, and that velocity can be determined by differentiating the position function $s = f(t)$.

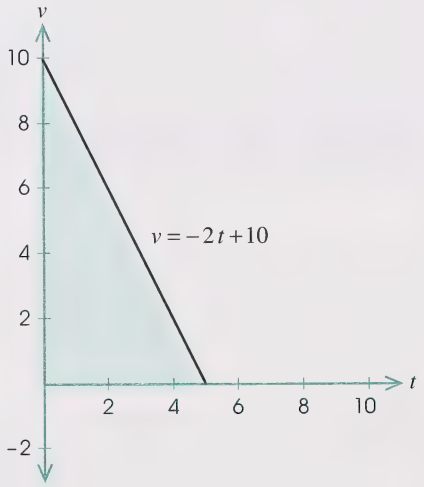
That is, $v = \dfrac{ds}{dt}$.

It should now be possible, by taking the antiderivative, to work backwards from velocity to position.

Example 1

Anna reaches a velocity of 10 m/s at the bottom of the hill. From that point on, she coasts in a straight line. If her velocity (in m/s) while she coasts is given by $v = -2t + 10$, what is Anna's position s (in metres), measured from the bottom of the hill, at any time t (in seconds), where $t \geq 0$? How far will she coast?

Solution



The velocity function $v = -2t + 10$, where $t \geq 0$, may be written as a differential equation.

$$\frac{ds}{dt} = -2t + 10$$

Solve the differential equation by finding the indefinite integral.

$$s = \int (-2t + 10)\, dt + C$$
$$= -t^2 + 10t + C$$

Evaluate the constant of integration C by substituting the initial conditions into the equation.

Since Anna's position is measured from the bottom of the hill, $s = 0$ at $t = 0$.

$$0 = -0^2 + 10(0) + C$$
$$C = 0$$

$\therefore\ s = -t^2 + 10t$, where $t \geq 0$

Now, Anna will continue to slide until her velocity is zero.

When $v = 0$, $v = -2t + 10$

$$0 = -2t + 10$$
$$2t = 10$$
$$t = 5$$

Anna continues to slide for 5 s.

Next, find Anna's position at $t = 5$ s.

$$s = -52 + 10(5)$$
$$= -25 + 50$$
$$= 25$$

At $t = 5$ s, Anna will be at 25 m.

Since $s = 0$ m at $t = 0$ s and $s = 25$ m at $t = 5$ s, the change in Anna's position (or displacement) is 25 m. In this example, because Anna always travels in the same direction, the distance she coasts before coming to rest will be the same as the displacement.

Also, note that the distance travelled equals the shaded area between the velocity function and the x-axis.

This area may be calculated using the formula for a triangle:

$$A = \frac{1}{2} \text{ base} \times \text{height}$$
$${}_0A_5 = \frac{1}{2}(5)(10)$$
$$= 25$$

The area may also be obtained from the definite integral.

$${}_0A_5 = \int_0^5 (-2t + 10)\, dt$$
$$= \left[-t^2 + 10t\right]_0^5$$
$$= \left[-5^2 + 10(5)\right] - \left[-0^2 + 10(0)\right]$$
$$= 25$$

Example 1 illustrates two principles.

- The position s of an object is the solution to the differential equation formed from the velocity function $v = f(t)$.

  Therefore, if $v = \frac{ds}{dt} = f(t)$, then $s = \int f(t)\ dt + C$. Normally, C is evaluated from initial conditions.

- The distance travelled is the area between the graph of the velocity function and the x-axis.

Work through another example.

Example 2

A rock is tossed upward at 10 m/s from the top of a building 15 m above the ground. The velocity (in m/s) of the rock at any time t (in seconds) is given by $v = -10t + 10$, where $t \geq 0$. Determine the rock's height h (in metres) at any time t. Determine the total distance the rock travels before it strikes the ground.

Solution

Because $v = -10t + 10$, where $t \geq 0$, solve $\frac{dh}{dt} = -10t + 10$.

$$h = \int (-10t + 10)\ dt + C$$
$$= -5t^2 + 10t + C$$

Since $h = 15$ at $t = 0$, calculate C.

$$15 = -5(0)^2 + 10(0) + C$$
$$C = 20$$

$\therefore\ h = -5t^2 + 10t + 15$, where $t \geq 0$

Since the rock rises and then falls, you must find the stationary point of the position function to determine how far the rock travels.

At the stationary point, $\frac{dh}{dt} = v = 0$.

$$-10t + 10 = 0$$
$$t = 1$$

The rock reaches a maximum height at $t = 1$ s.

When $t = 1$, $h = -5(1)^2 + 10(1) + 15$
$= 20$

In the first second, the rock rises 5 m from the top of the building 15 m above the ground to its maximum height of 20 m. The rock then falls 20 m to the ground. Thus, the total distance the ball travels is $5\ \text{m} + 20\ \text{m} = 25\ \text{m}$.

As in Example 1, the total distance can be found using definite integrals and evaluating the area between the velocity function and the t-axis. First, you must determine how long the rock is in the air.

When the rock strikes the ground, $h = 0$.

$$0 = -5t^2 + 10t + 15$$

$$-5\left(t^2 - 2t - 3\right) = 0$$

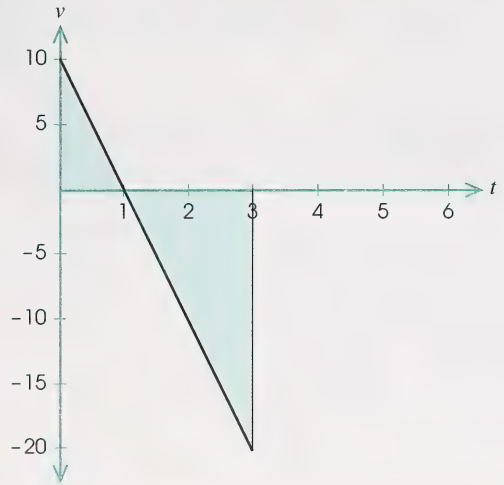
$$-5(t-3)(t+1) = 0$$

$$t - 3 = 0 \text{ or } t + 1 = 0$$

$$t = 3 \qquad t = -1$$

Because time is non-negative, $t = 3$ is the total time the rock is in the air.

Graph the velocity function for $0 \le t \le 3$.



$$\text{Total Area} = {}_0A_1 + {}_1A_3$$

$$= \int_0^1 (-10t + 10)\, dt + \left| \int_1^3 (-10t + 10\, dt) \right|$$

$$= \left[-5t^2 + 10t\right]_0^1 + \left|-5t^2 + 10t\right|_1^3$$

$$= \left[-5(1)^2 + 10(1)\right] - \left[-5(0)^2 + 10(0)\right]$$

$$+ \left|\left[-5(3)^2 + 10(3)\right] - \left[-5(1)^2 + 10(1)\right]\right|$$

$$= (-5 + 10) - (0 + 0) + \left|(-45 + 30) - (-5 + 10)\right|$$

$$= 5 + \left|-15 - 5\right|$$

$$= 5 + \left|-20\right|$$

$$= 5 + 20$$

$$= 25$$

The total area is 25; therefore, the distance the rock travels is 25 m.

If you had not taken into account the fact that the velocity graph crossed the t-axis at $t = 1$, and had simply evaluated $\int_0^3 (-10t + 10)\, dt$, then the result would have been

$5 \text{ m} - 20 \text{ m} = -15 \text{ m}$.

The result -15 m is the **total displacement** rather than the total distance. Do you agree that the rock ends up 15 m below its initial position?

1. Find the position function from the velocity function and the given initial conditions. Assume $t \geq 0$.

   a. $v = (1-2t)^3$, and $s = 1$ when $t = 0$

   b. $v = (t-1)^{\frac{1}{2}}$, and $s = 4$ when $t = 2$

   c. $v = 2\cos 3t$, and $s = -1$ when $t = \frac{\pi}{6}$

2. The velocity v (in m/s) of an object moving in a straight line is given by $v = 3t^2 - 6t$. How far does it travel from $t = 0$ s to $t = 3$ s? Assume that $s = 0$ m when $t = 0$ s.

3. A sandbag is released at an altitude of 100 m from a balloon which is rising at 2 m/s. The velocity v of the sandbag at time t seconds is given by $v = 2 - 10t$. When will the bag strike the ground?

4. The brakes are applied in a car travelling downhill at 40 m/s. The velocity v at time t seconds after the brakes are applied is given by $v = 40 - 5t^{\frac{3}{2}}$. How far will the car travel before coming to rest?

Check your answers by turning to the Appendix.

Velocity functions vary from situation to situation. Can you think of factors that would change the velocity function of a person sliding down a snow-covered hill on a carpet?

Activity 2: Velocity from Acceleration

Part of the excitement of sailing is the feel of the wind and the sensations of speed and acceleration.

In Module 4, you calculated the acceleration a, of an object travelling along a straight line, by differentiating the velocity function $v = f(t)$.

$$a = \frac{dv}{dt} = f'(t)$$

Knowing that integration is the inverse operation of differentiation, you should now be able to reverse the process and find a velocity function by solving a differential equation.

Therefore, if $\frac{dv}{dt} = a$, then $v = \int a\,dt + C$.

You will also recall that acceleration is the second derivative of the position function s.

If $s = f(t)$, then $a = \frac{d^2 s}{dt^2} = f''(t)$.

Therefore, given the acceleration function, you can determine the position function by integrating twice; first to find velocity, and second to find position.

Example 1

Kaven, an avid skier, is accelerating down a run at $2\ \text{m/s}^2$. His position d measured from the top of the hill, is 10 m at $t = 0$ s. If Kaven's initial speed v (at $t = 0$ s) as he begins to accelerate is 1 m/s, determine the following:

- his speed at any time t
- his location relative to the top of the hill at any time
- the distance skied by the time his speed reaches 11 m/s

Solution

Assuming Kaven skis in a straight line (because acceleration is positive), speed and velocity are the same.

$$\begin{aligned} v &= \int a\,dt + C_1 \\ &= \int 2\,dt + C_1 \\ &= 2t + C_1 \end{aligned}$$

Subscripts are used to distinguish between the constants in the velocity and displacement functions.

Use the initial conditions to find C_1.

Now, $v = 1$ m/s when $t = 0$ s.

$$\begin{aligned} 1 &= 2(0) + C_1 \\ C_1 &= 1 \end{aligned}$$

Therefore, Kaven's speed at any time is $v = 2t + 1$.

$$\begin{aligned} d &= \int v\, dt \\ &= \int (2t+1)\, dt + C_2 \\ &= t^2 + t + C_2 \end{aligned}$$

When $t = 0$ s, $d = 10$ m.

$$\begin{aligned} 10 &= 0^2 + 0 + C_2 \\ C_2 &= 10 \end{aligned}$$

Therefore, Kaven's location relative to the top of the hill is $d = t^2 + t + 10$.

From the velocity function, determine when he is skiing at 11 m/s.

$$\begin{aligned} v &= 2t + 1 \\ 11 &= 2t + 1 \\ t &= 5 \text{ s} \end{aligned}$$

Substitute into the position function to determine Kaven's location.

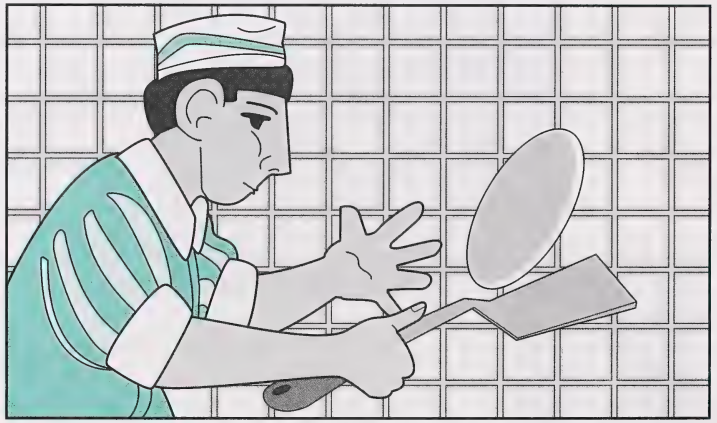
$$\begin{aligned} d &= 5^2 + 5 + 10 \\ &= 40 \end{aligned}$$

Kaven is 40 m from the top of the hill when he reaches a velocity of 11 m/s. Since he started 10 m from the top, the distance he skied was 30 m.

In Example 1, position, velocity, and acceleration were all measured in the same direction; therefore, you represented them as positive quantities.

However, if opposite directions were involved, the usual conventions would apply. **Upwards** or **to the right** are **positive**; **downwards** or **to the left** are **negative**. Look at Example 2.

Example 2



Bruce flips a pancake straight upwards at a speed of 2 m/s. If the pancake leaves the pan 1.5 m above the floor, to what height does it rise? Assume that the acceleration due to gravity is -10 m/s^2.

Solution

Here, position and initial velocity are opposite in direction to the acceleration. The acceleration due to gravity acts downward and is assigned a negative sign.

$$\begin{aligned} v &= \int a\, dt \\ &= \int (-10)\, dt + C_1 \\ &= -10t + C_1 \end{aligned}$$

When $t = 0$ s, $v = +2$ m/s.

Upwards is positive!

$$\begin{aligned} 2 &= -10(0) + C_1 \\ C_1 &= 2 \end{aligned}$$

$$\therefore\ v = -10t + 2$$

The pancake reaches a maximum height when $v = 0$ m/s.

$$\begin{aligned} 0 &= -10t + 2 \\ t &= 0.2 \end{aligned}$$

To determine the height to which the pancake rises, determine the position function h.

$$\begin{aligned} h &= \int v\, dt + C_2 \\ &= \int (-10t + 2)\, dt + C_2 \\ &= -5t^2 + 2t + C_2 \end{aligned}$$

Evaluate C_2 from initial conditions.

At $t = 0$ s, $h = 1.5$ m.

$$\begin{aligned} 1.5 &= -5(0)^2 + 2(0) + C_2 \\ C_2 &= 1.5 \end{aligned}$$

$$\therefore\ h = -5t^2 + 2t + 1.5$$

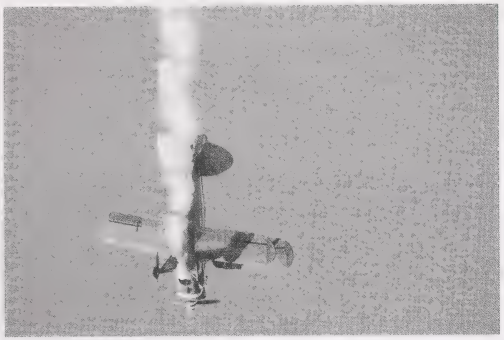
When $t = 0.2$ s, $h = -5t^2 + 2t + 1.5$

$$\begin{aligned} &= -5(0.2)^2 + 2(0.2) + 1.5 \\ &= 1.7 \end{aligned}$$

The pancake rises to a height of 1.7 m. That's just 1.7 m − 1.5 m = 0.2 m or 20 cm above the pan.

In the questions that follow, when applicable, use $-10\ \text{m/s}^2$ for the acceleration due to gravity.

1. A bolt falls off a stunt plane flying straight downward at a speed of 100 m/s. If the plane was at an altitude of 500 m when this occurred, at what speed does the bolt strike the ground?



2. A ball is tossed vertically upward at a speed of 20 m/s. If the ball was released 2 m above the ground, how high does it rise?

3. On an icy road, a car travelling at 30 m/s brakes and decelerates at $3\ \text{m/s}^2$. What is the stopping distance?

Check your answers by turning to the Appendix.

If you have studied physics, you probably memorized the formulas from **kinematics** for determining distance, velocity, and acceleration. One of the situations in which you likely used those formulas was solving problems involving acceleration due to gravity. Integral calculus gives you a method for deriving those formulas quickly and easily. Finding solutions to problems like these may have motivated Newton to develop calculus.

Example 3

Develop the following kinematic formulas for the velocity v and position d at any time t (where $t \geq 0$) of an object accelerating at a constant rate a and travelling in a straight line. Assume that at $t = 0$, $d = d_0$ and $v = v_0$.

- $v = at + v_0$
- $d = \frac{1}{2}at^2 + v_0 t + d_0$

Solution

Develop the velocity formula.

$$v = \int a\,dt + C_1$$
$$= at + C_1$$

Remember, a is a constant.

Evaluate C_1 using the initial conditions $v = v_0$ when $t = 0$.

$$v_0 = a(0) + C_1$$
$$C_1 = v_0$$

$$\therefore\ v = at + v_0$$

Develop the position formula.

$$d = \int v\,dt$$
$$= \int (at + v_0)\,dt + C_2$$
$$= \frac{1}{2}at^2 + v_0 t + C_2$$

When $t = 0$, $d = d_0$.

$$d_0 = \frac{1}{2}a(0)^2 + v_0(0) + C_2$$
$$C_2 = d_0$$

$$\therefore\ d = \frac{1}{2}at^2 + v_0 t + d_0$$

Do questions 4 and 5 using the formulas you derived in Example 3.

4. Show that the kinematic formulas could have been used to obtain the equations for velocity and position in Example 2.

5. For linear motion, derive the formula $v^2 + {v_0}^2 + 2as$, where

v = velocity at time t
v_0 = initial velocity
a = constant acceleration
s = displacement at time t
$= (\text{position at time } t) - (\text{initial position})$
$= d - d_0$

Check your answers by turning to the Appendix.

All the questions you have done so far have involved constant acceleration. There is no reason to restrict the procedures outlined in Examples 1 and 2. Problems involving variable acceleration are approached the same way. Of course, you cannot use the formulas from Example 3 since a was a constant.

6. An object is travelling in a straight line. Its acceleration at time t is $a = -2t$. Find the velocity function v and the position function s. At $t = 0$, $v = 2$, and $s = 8$.

7. A rocket is accelerating vertically from rest at $(3t+2)$ m/s^2. How far will it travel in the first three seconds?

Check your answers by turning to the Appendix.

When you apply the brakes in a car, does it decelerate uniformly? What factors would affect stopping distance?

Activity 3: Harmonic Motion

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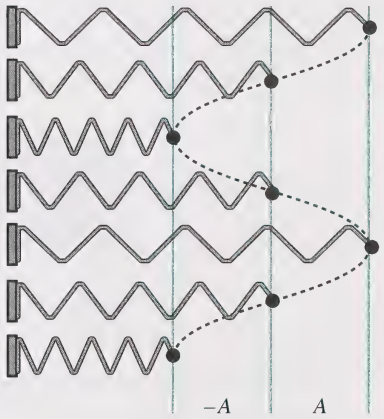
What does playing the violin, releasing a bow cord, causing a mass attached to a spring to oscillate, setting a pendulum in motion, and watching the shadow cast by a rock being swung in a circle have in common? Each is an example of harmonic motion.

Harmonic motion, like the motion described in the previous activities, involves relationships among position, displacement, velocity, and acceleration. As you will see, this relationship can be described, in part, by the displacement equation $x = A\cos(bt+c)$.

Consider a mass m resting on a frictionless surface and attached to a spring. Examine the following diagram.

Expansion and Compression of a Spring



If you pull the mass x units to the right, the spring exerts a force F that pulls the mass to the left, trying to restore it to its **equilibrium** or **rest position**. Similarly, the restorative force is to the right when the spring is compressed. The force F is proportional to the displacement x, but opposite in direction. For instance, if you pull the mass twice as far to the right, the restorative force will be twice as large to the left.

Therefore, $F \propto x$ or $F = -kx$, if k is positive.

If the spring is stretched A units to the right and released, the mass will oscillate as shown in the spring diagram. The acceleration a at time t of the mass will depend on the displacement x.

Using Newton's Second Law, $F = ma$, $ma = -kx$ or $a = -\frac{k}{m}x$.

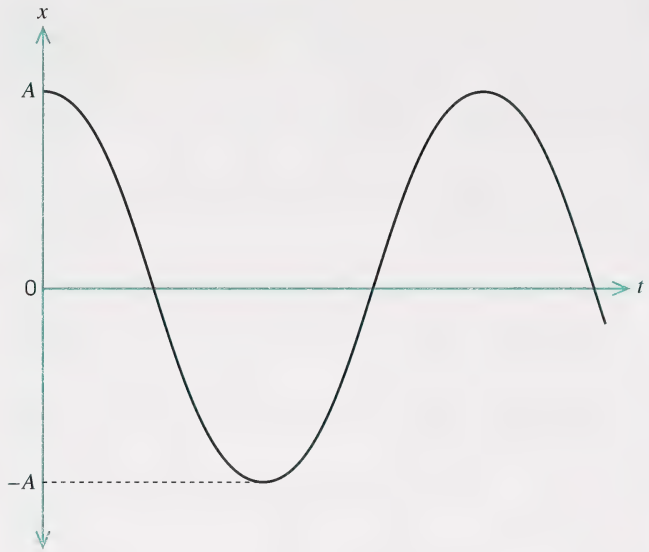
But, you know $a = \frac{d^2x}{dt^2}$.

$$\therefore \frac{d^2x}{dt^2} = -\frac{k}{m}x \text{ or } \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Any motion that is described by a differential equation of the form $\frac{d^2x}{dt^2} + Kx$ for some constant K is called **simple harmonic motion**. The vibrating spring, the string of a guitar, a simple pendulum, and an alternating current are a few examples of simple harmonic oscillators.

In the example of the expansion and compression of a spring, notice that the acceleration would be a maximum when the displacement is a maximum. The velocity at that instant is zero. When the mass passes through the rest position, the displacement and acceleration are zero, but the mass is travelling the fastest.

If you graphed consecutive positions of the mass over time, it would trace out a sine or cosine graph, as shown in both the spring diagram and in the following graph.



You will see that the solution to a differential equation of the form $\frac{d^2x}{dt^2} + Kx = 0$, for some constant K is a displacement (or position) function of the type $x = A\cos(kx + c)$.

Example 1

Show that $x = A\cos(kx + c)$ is the solution to a second-order differential equation modelling simple harmonic motion.

Solution

Differentiate the displacement equation twice to determine the acceleration.

$$x = A\cos(kx + c) \qquad \frac{dx}{dt} = -A[\sin(kx + c)]k$$
$$= -Ak\sin(kx + c)$$

$$\frac{d^2x}{dt^2} = -Ak[\cos(kx + c)]k$$
$$= -Ak^2\cos(kx + c)$$

Notice that the acceleration is a multiple of $x = A\cos(kx + c)$, the original displacement function.

$$\frac{d^2x}{dt^2} = -k^2[A\cos(kx + c)]$$
$$= -k^2x$$

or $\frac{d^2x}{dt^2} + k^2x = 0$

$\frac{d^2x}{dt^2} + k^2x = 0$ is a second-order differential equation modelling simple harmonic motion.

1. A mass hangs from a spring. A spring supporting the mass is stretched a further 4 cm and released. The displacement function is $x = -4\cos(\pi t)$ at time t.

a. Determine the maximum and the minimum displacement, and the times when they occur.

b. What is the period of the motion?

c. When is the velocity a maximum?

d. Determine the second-order differential equation which models the motion.

2. The displacement s (in metres) of a pendulum bob from the vertical is given by $s = 0.04\cos\left(\frac{\pi t}{3}\right)$.

a. Determine the maximum and the minimum displacement, and the times when they occur.

b. What is the period of the motion?

c. When is the velocity a maximum?

d. Determine the second-order differential equation which models the motion.

3. Show that each of the following is a solution to a second-order differential equation which models simple harmonic motion.

a. $x = \sin\ (3t - 2)$

b. $x = 2 \sin\ 3t + 5 \cos\ 3t$

Check your answers by turning to the Appendix.

As you have seen, the displacement function may be expressed in a variety of ways. The following represent three choices.

- $x = A \cos\ (kx + c)$
- $x = A \sin\ (kx + c)$
- $x = A \sin\ kx + B \cos\ kx$

Identities may be used to transform one form into another form. For instance, if you use $\cos\ (\alpha + \beta) = \cos\ \alpha\ \cos\ \beta - \sin\ \alpha\ \sin\ \beta$,

then $A \cos\ (kx + c) = A(\cos\ kx\ \cos\ c - \sin\ kx\ \sin\ c)$

$= A \cos\ c\ \cos\ kx - A \sin\ c\ \sin\ kx$

$= A_1 \cos\ kx + B_1 \sin\ kx$

In the last part of this activity, you will work backwards from the second-order differential equation or the differential equation describing velocity to find the displacement function.

Example 2

The velocity v of the midpoint of a vibrating cello string at time t is given by $v = 0.256\pi \cos\ 256\pi t$. If the displacement $x = 0$ m at $t = 0$ s, find the displacement x and the acceleration a at any time t. What is the period and frequency of the motion?

Solution

Since $v = 0.256\pi \cos\ 256\pi t$, $x = \int v\ dt + C$

$= \int 0.256\pi \cos\ 256\pi t\ dt + C$

$= \frac{0.256\pi}{256\pi} \sin\ 256\pi t + C$

$= 0.001 \sin\ 256\pi t + C$

Since $x = 0$ at $t = 0$, $0 = 0.001 \sin\ 256\pi(0) + C$

$0 = 0.001 \sin\ 0 + C$

$C = 0$

Therefore, $x = 0.001 \sin\ 256\pi t$.

$a = 0.256\pi(-\sin\ 256\pi t)(256\pi)$

$= -256^2\ \pi^2\ (0.001 \sin\ 256\pi t)$

$= -256^2\ \pi^2\ x$

$\therefore\ \frac{d^2x}{dx^2} = -256^2\ \pi^2\ x$

$$\therefore \frac{d^2x}{dx^2} + 256^2\,\pi^2 x = 0$$

To find the period of the motion, use the formula from Mathematics 30. The period of $\sin b\phi$ is $\frac{2\pi}{b}$, the period of $x = 0.001 \sin 256\pi t$ is $\frac{2\pi}{256\pi}$ or $\frac{1}{128}$ s. If one cycle is completed in $\frac{1}{128}$ s, then the frequency is 128 cycles/s or 128 Hz.

4. A swimmer floating in the water executes simple harmonic motion as the waves pass underneath. The vertical velocity v at time t of the swimmer is given by $v = \pi \cos 2\pi t$. If the displacement y is 0 m at $t = 0$ s, find the displacement function, the maximum height to which the swimmer rises, and the acceleration at any time t.

5. Find the displacement function $y = f(t)$, if $\frac{d^2y}{dt^2} = \sin t$, $\frac{dy}{dt} = 3$, and $y = 0$ at $t = 0$.

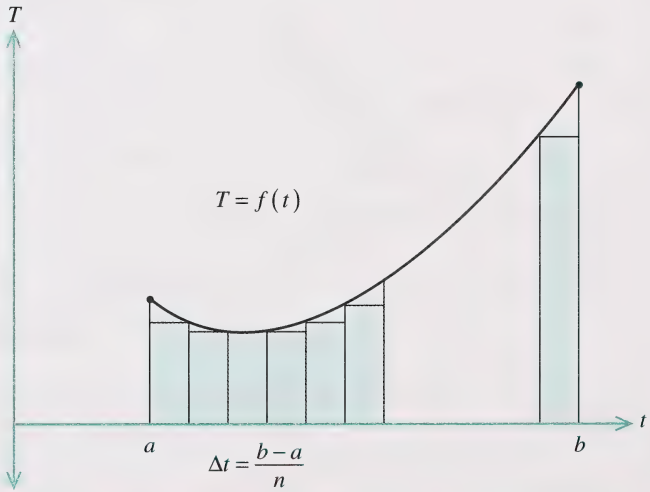
Check your answers by turning to the Appendix.

The mathematics of musical harmony extends beyond a single vibrating string. Three strings of lengths a, b, and c produce harmony if those lengths form a harmonic progression:

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

Activity 4: The Mean Value of a Function

When a newspaper reports the mean temperature for a given twenty-four-hour period, that temperature is normally calculated by averaging the daily high and low. If the maximum were 23°C and the minimum were 11°C, the mean would be $\frac{23°C + 11°C}{2} = 17°C$. However, it certainly would be possible to record the temperature every quarter hour and divide the resulting sum by 96. Or, the mean could be the average of temperatures taken every minute. If the temperature were taken continuously over a twenty-four-hour period, how would the average be calculated?



The preceding diagram represents a continuous change in temperature T between times $t = a$ and $t = b$. The time interval $(b - a)$ has been subdivided into n subintervals, each of width Δt, where $\Delta t = \frac{b-a}{n}$.

If Δt represents a sufficiently short period of time, the temperature is essentially constant in that time interval. The mean temperature would be approximated by $\frac{f(t_1) + f(t_2) + f(t_3) + \ldots + f(t_n)}{n}$, where t_1 is the time in the first subinterval for which an inscribed rectangle can be drawn, t_2 is the time in the second subinterval for which an inscribed rectangle can be drawn, and so on; $f(t_1)$, $f(t_2)$, and so on are the corresponding temperatures.

$$T_{\text{mean}} = \frac{\sum_{i=1}^{n} f(t_i)}{n}$$

$$\doteq \frac{\sum_{i=1}^{n} f(t_i)}{\frac{b-a}{\Delta x}}$$

$$\doteq \frac{1}{b-a} \sum_{i=1}^{n} f(t_i) \bullet \Delta t$$

The approximation would become more accurate as $\Delta t \to 0$.

$$T_{\text{mean}} = \frac{1}{b-a} \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(t_i) \bullet \Delta t$$

The limit defines the definite integral of $f(t)$ from $t = a$ to $t = b$. Therefore, the mean temperature can be defined as follows:

$$T_{\text{mean}} = \frac{1}{b-a} \int_a^b f(t)\, dt$$

This expression can be viewed as the area under the curve divided by the interval length $(b - a)$. This provides a way of determining the **mean (or average) value of a function**, $y = f(x)$, with respect to x, particularly if that function is continuous over the interval $[a, b]$.

$$(y_{\text{ave}})_x = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Example 1

The velocity v at time t of an object dropped from rest is given by $v = gt$, where g is the acceleration due to gravity. Find the mean or average velocity with respect to t between the times $t_1 = 0$ and t_2.

Solution

Apply the formula $(y_{\text{ave}})_x = \frac{1}{b-a} \int_a^b f(x)\, dx$.

In this example, the given function is $v = gt$, $a = 0$, and $b = t_2$.

$$\begin{aligned}
\left(v_{\text{ave}}\right)_t &= \frac{1}{t_2 - 0}\int_0^{t_2} v\,dt \\
&= \frac{1}{t_2}\int_0^{t_2} gt\,dt \\
&= \frac{1}{t_2}\left[\frac{gt^2}{2}\right]_0^{t_2} \\
&= \frac{1}{t_2}\left(\frac{gt_2{}^2}{2} - \frac{g(0)^2}{2}\right) \\
&= \frac{1}{t_2}\left(\frac{gt_2{}^2}{2}\right) \\
&= \frac{1}{2}gt_2
\end{aligned}$$

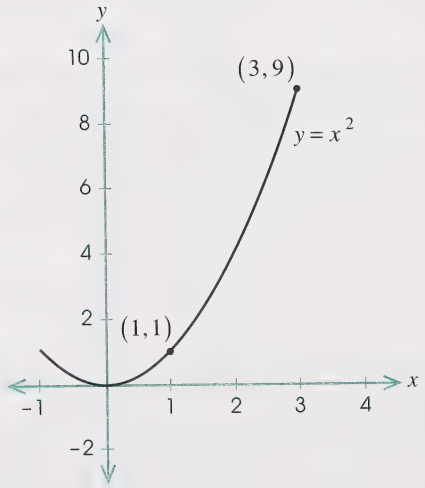
Note: The initial velocity is equal to $g(0) = 0$, and the final velocity is equal to gt_2. The average velocity is their arithmetic mean.

Is the average value of the function always the mean of the first and last values of the function?

Example 2

Find the average value of y, with respect to x, for the function $y = x^2$, between $x = 1$ and $x = 3$.

Solution



Apply $\left(y_{\text{ave}}\right)_x = \frac{1}{b-a}\int_a^b f(x)\,dx$, where $f(x) = x^2$, $a = 1$, and $b = 3$.

$$\therefore \left(y_{\text{ave}}\right)_x = \frac{1}{3-1}\int_1^3 x^2\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_1^3$$

$$= \frac{1}{2}\left[\frac{3^3}{3} - \frac{1^3}{3}\right]$$

$$= \frac{1}{2}\left[\frac{27}{3} - \frac{1}{3}\right]$$

$$= \frac{1}{2}\left[\frac{26}{3}\right]$$

$$= \frac{13}{3}$$

Compare your answer with $\frac{f(3)+f(1)}{2}$.

$f(1) = 1^2 = 1$

$f(3) = 3^2 = 9$

The mean of the functional values at $x = 1$ and $x = 3$ is $\frac{1+9}{2} = 5$. As you can see, this value is **not** the same as the average value of y with respect to x. The only time they are equal is when the function is linear, as in Example 1.

1. Find the average value with respect to x of $f(x) = 3x^2 + 2x$ between $x = 0$ and $x = 1$. Relate your answer to the graph of the function.

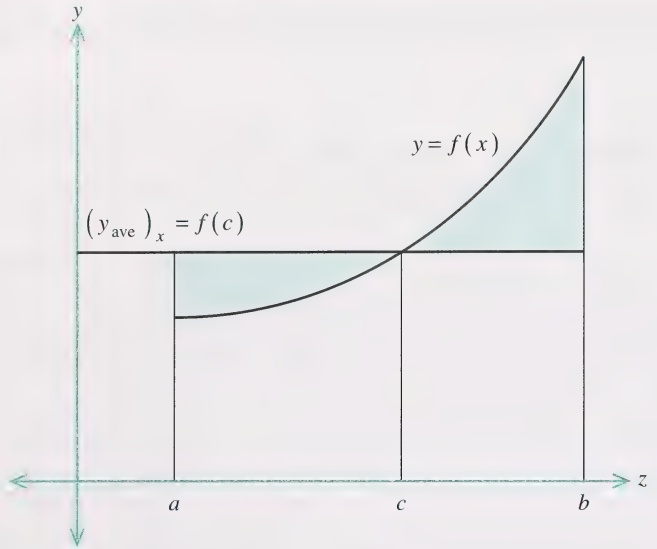
2. In Example 1, you saw that for an object falling from rest, its average velocity with respect to t was $\frac{1}{2}gt_2$. The speed the object attains may also be expressed in terms of the distance d it falls; that is, – $v = \sqrt{2gd}$. Determine the average velocity with respect to d between those distances at which $t = 0$ and $t = t_2$; that is, $d = 0$ and $d = \frac{1}{2}g{t_2}^2$. Is the answer the same as in Example 1?

3. Real numbers between 0 and 10 are selected at random. What value does the mean of their square roots approach?

4. The temperature T of a beaker of water being heated rises uniformly from 20°C to 40°C over a 50 s interval. What is the mean temperature with respect to time t? How is your answer related to the temperatures given in the problem? Why?

Check your answers by turning to the Appendix.

How is the mean or average value of a continuous function $\left(y_{\text{ave}}\right)_x = \frac{1}{b-a}\int_a^b f(x)\,dx$ related to its graph?



The definite integral represents the area between the curve and the x-axis, from $x = a$ to $x = b$. When you divide this area by its **base** $(b - a)$, you obtain the **altitude** of a rectangle with the same area. That is, the mean value of a function is the altitude of a rectangle with the same area and the same base as the area described by the definite integral. In the figure, the shaded area above the top of the rectangle is the same as the shaded area below the top of the rectangle!

For a continuous function defined on the interval $[a, b]$, there exists a real value c, such that $a \le c \le b$ and $f(c) = (y_{\text{ave}})_x$.

Example 2

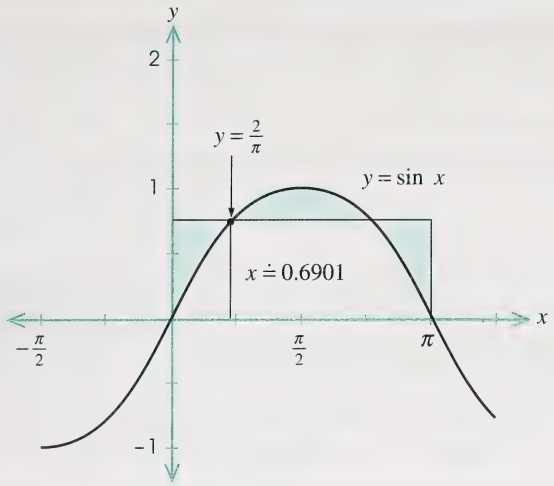
Find the mean value with respect to x for $y = \sin\ x$, between $x = 0$ and $x = \pi$. Sketch the function and the rectangle which has the altitude of the mean y-value.

Solution

Apply $(y_{\text{ave}})_x = \frac{1}{b-a}\int_a^b f(x)\ dx$ where $f(x) = \sin\ x$, $a = 0$, and $b = \pi$.

$$
\begin{aligned}
(y_{\text{ave}})_x &= \frac{1}{\pi - 0}\int_0^{\pi} \sin\ x\ dx \\
&= \frac{1}{\pi}[-\cos\ x] \\
&= \frac{1}{\pi}[-\cos\ \pi - (-\cos\ 0)] \\
&= \frac{1}{\pi}[-(-1) - (-1)] \\
&= \frac{2}{\pi}
\end{aligned}
$$

To find the x-value in the interval $[0, \pi]$ that gave you this y-value, you must solve $\sin\ x = \frac{2}{\pi} \doteq 0.6901$.



5. Find the mean value of the following with respect to x. Sketch the function and the rectangle which has the altitude of the mean y-value.

a. $f(x) = \sqrt{2x+1}$, where $x = 0$ and $x = 12$

b. $f(x) = \sin\ x$, where $x = 0$ and $x = 2\pi$

Check your answers by turning to the Appendix.

Sketching functions is like discussing the weather. Knowledge of the mean values is important, but so is knowledge of the extremes!

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you had a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

You may wish to review the video *Applications of Integrals to Area and Motion* from the *Catch 31* series, ACCESS Network. The first half of this video reviews procedures for calculating the area between the graph of a function and the x-axis, and for calculating the area between two curves. The second half of the video summarizes techniques for determining velocity from acceleration, and determining displacement from velocity. As well, the relationships between the graphs of velocity and displacement are described. This video is available from the Learning Resources Distributing Centre.

Many students confuse the concepts of **position**, **displacement**, and **distance**.

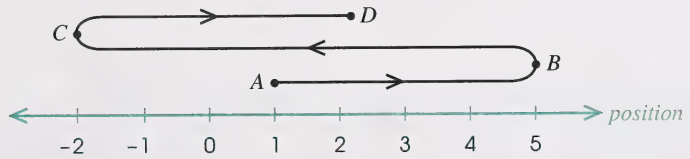
Position and displacement are vector quantities; that is, they are represented by both magnitude and direction. For an object moving in a straight line, the position of that object is described relative to an origin or zero position. Position will be either positive or negative according to the usual conventions. For an object in free fall, for example, the origin is usually located at ground level, and a positive position is above the ground.

Displacement describes a change in position between two times. For linear motion, if the change in position is in a positive direction (to the right or upward), displacement is positive; if the change in position is in a negative direction, displacement is negative.

Distance, on the other hand, describes how far an object has travelled. Distance is non-negative. For an object that changes direction as it moves in a straight line, distance travelled will differ from position, and position from displacement.

Consider the following example.

Example



The preceding diagram represents successive positions, at one-second intervals, of an object travelling in a straight line.

At $t = 0$ s, the position of the object is at Point A, or at $+1$ units, measured from the origin.

At $t = 1$ s, the object is at Point B, or at $+5$ units, measured from the origin.

At $t = 2$ s, the object is at Point C, or at -2 units, measured from the origin.

At $t = 3$ s, the object is at Point D, $+2$ units measured from the origin.

Find the displacement of the object between $t = 0$ s and $t = 3$ s. Determine the total distance travelled between $t = 0$ s and $t = 3$ s.

Solution

The object at $t = 0$ s is at $+1$. At $t = 3$ s, its position is $+2$.

$$\begin{aligned}\text{displacement} &= \text{final position} - \text{initial position} \\ &= (+2) - (+1) \\ &= +1\end{aligned}$$

Therefore, the displacement is from $+1$ to $+2$, a total of one unit to the right.

The total distance travelled between those times is the sum of the distances from A to B, B to C, and C to D.

$$\begin{aligned}\text{distance} &= 4 + 7 + 4 \\ &= 15\end{aligned}$$

The total distance travelled is 15 units.

As you can see, position, displacement, and distance are interrelated and are all functions of time. If, for instance, an object is thrown upward at 10 m/s from a height of 2 m, the position function is $h = -5t^2 + 10t + 2$. When the object hits the ground, its position would be $h = 0$ m, its displacement from $t = 0$ s would be -2 m, and the distance travelled would be the distance it rises (5 m) plus the distance it falls (7 m) or 12 m altogether.

Some texts will call the preceding equation a displacement function, where displacement at time t is always measured from the origin. In that case, position and displacement would be synonymous. You should always check from the context of the problem, what the author means by displacement.

The following questions refer to the example in the Extra Help.

1. What is the displacement from A to C?
2. What is the distance the object travels from A to C?
3. What is the object's average velocity from A to C?
4. What is the object's average speed from A to C?

Check your answers by turning to the Appendix.

Enrichment

Another application of the definite integral is the concept of **work** in physics. Work is done when a force F acts through a distance s. The amount of work W is defined as the product of that force and distance.

$$W = Fs$$

If the force is measured in newtons and distance in metres, work is in joules (J).

1 joule = 1 newton metre

$1 \text{ J} = 1 \text{ N} \cdot \text{m}$

Example 1

A litre container of milk weighs 10 N. How much work is done when that container is lifted vertically 0.5 m from the shelf in the refrigerator up to the counter?

Solution

$W = Fs$, where $F = 10$ N and $s = 0.5$ m.

$= (10 \text{ N})(0.5 \text{ m})$

$= 5 \text{ N} \cdot \text{m}$

$= 5 \text{ J}$

The amount of work done is 5 J or $5 \text{ N} \cdot \text{m}$.

This example only involved a constant force F. What would you do if the problem involved a variable force (a force that was a function of position s)?

$$F = f(s)$$

If the force varies, then the amount of work done is the average force times the distance.

Consider a force, $F = f(s)$, applied in a given direction from $s = a$ to $s = b$, where $a \le b$. Then, as in Activity 4,

$$\left(F_{\text{ave}}\right)_s = \frac{1}{b-a}\int_a^b f(s)\, ds.$$

Now, the distance through which this force acts is $(b-a)$.

$$\therefore \text{ work} = \left(F_{\text{ave}}\right)_s \times (b-a)$$

$$= \left[\frac{1}{b-a}\int_a^b f(s)\, ds\right](b-a)$$

$$W = \int_a^b f(s)\, ds$$

Example 2

A force of 10 N is required to compress a 10-cm-long spring to 8 cm. If the force F required to compress the spring varies directly as the amount the spring is compressed s, determine the force function $F = f(s)$. How much work is done in compressing that spring from a length of 8 cm to a length of 5 cm?

Solution

Find the force function.

$$\text{Force} \propto \text{amount compressed}$$

$$F \propto s$$

$F = ks$, where k is the constant of variation called the **spring constant**.

A compression from 10 cm to 8 cm is 0.02 m.

$$10 \text{ N} = k(0.02 \text{ m})$$

$$k = \frac{10}{0.02}$$

$$= 500$$

$$\therefore F = 500s$$

Next, apply the formula for work.

Remember, s = amount compressed.

When the spring is 8 cm in length, the compression $a = (10\ \text{cm} - 8\ \text{cm}) = 0.02\ \text{m}$.

When the spring is 5 cm in length, the compression $b = (10\ \text{cm} - 5\ \text{cm}) = 0.05\ \text{m}$.

$$\begin{aligned} W &= \int_a^b f(s)\ ds \\ &= \int_{0.02}^{0.05} (500s)\ ds \\ &= \left[250s^2\right]_{0.02}^{0.05} \\ &= 250\left[0.05^2 - 0.02^2\right] \\ &= 250(0.0025 - 0.0004) \\ &= 0.525 \end{aligned}$$

The work being done is 0.525 J.

Now try the following:

1. A bag of sand, originally weighing 1000 N, is being raised at a constant speed. As the bag is raised, sand runs out of the bag at a constant rate. When the bag is raised 10 m, it weighs 600 N. What is the work done in raising that bag 10 m?

2. For every metre of its length, a uniform chain weighs 30 N. A length of 10 m hangs from a window. How much work is done if 5 m of this chain is pulled through the window?

Check your answers by turning to the Appendix.

Conclusion

In this section you examined a number of applications of integration. In Activity 1 and Activity 2, you used the antiderivatives of acceleration and velocity functions to obtain velocity and displacement functions. You solved problems associated with position, velocity, and acceleration in the forms $v = f(t)$ or $a = f(t)$. You were able to use calculus to derive the kinematic equations.

In Activity 3, you determined the equations of velocity and acceleration in simple harmonic motion starting from the displacement equation $x = A\cos(bt + c)$. You also worked backwards from the second-order differential equation modelling harmonic motion to obtain velocity and displacement.

In the last activity, you calculated the mean value of a function over an interval, and you looked at several applications of that process in everyday situations.

To become more proficient in downhill skiing, novice skiers are wise to heed the advice of expert skiers. With practice, novice skiers gain confidence as the techniques taught become automatic. Unconsciously, skiers use the physics of the sport to their advantage. Needless to say, the relationships among force, acceleration, velocity, and distance, described in this section, are important factors skiers must consider when elevating themselves to the expert level.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Module Summary

In this module, integration was used in two fundamental processes. In Section 1, you used integration to find functions from their derivatives; that is, you determined antiderivatives and solutions to differential equations. In Section 2, you used integration as a process of summation to address the problem of finding areas bounded by curves. The notation used in integration is related to summation. The Fundamental Theorem of Calculus tied summation and antiderivatives together.

Section 3 dealt with additional applications of integration. Specifically, you found position functions from velocity, and velocity functions from acceleration. You then extended your investigation to include harmonic motion. Finally, you used integration as a way of determining the average value of a function.

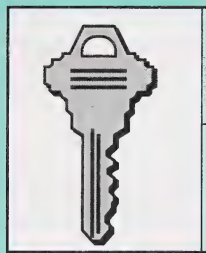
Think of the flow of traffic in a city neighbourhood. The relationships among the position, velocity, and acceleration functions of the vehicles may be described using both differential calculus and the techniques of integral calculus introduced in this module. If you pursue your study of calculus, you will investigate not only linear motion, but also motion in two or three dimensions.

Final Module Assignment

Assignment Booklet

You are now ready to complete the final module assignment.

APPENDIX



Glossary

Suggested Answers

Glossary

Acceleration: rate of change of velocity

Antiderivative: the original function obtained from its derivative

Circumscribed: drawn around a given figure

Comparison techniques: an integration technique based on "guess and test"

The "guess" is refined when the derivative of that "guess" is compared to the original integrand.

Constant of integration C: a constant that is added to the primitive, resulting from the integration process, to account for all possible antiderivatives

Definite integral: geometrically, the area above the x-axis minus the area below the x-axis bounded by a function $y = f(x)$ defined on the interval $[a, b]$ and the x-axis

If the primitive of $y = f(x)$ exists, then

$\int_a^b f(x)\,dx = F(b) - F(a)$.

Differential equation: an equation consisting of no more than two variables, and the first or higher-order derivatives of one of those variables with respect to the other

Displacement: the vector quantity representing change in position

Distance: how far an object travels in a given time, without regard to direction

Equilibrium: the rest position of a vibrating or oscillating object

Family of curves: a set of curves that can be obtained from the same condition, such as a common derivative

Fundamental Theorem of Calculus: the relationship between definite integrals and antiderivatives

$$ {}_aA_b = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} f(c_i)\Delta x = \int_a^b f(x)dx = F(b) - F(a) $$

Harmonic motion: periodic motion, such as that of a pendulum, that satisfies the second-order differential equation $y'' - ky = 0$

Indefinite integral: any function for which the given derivative is correct

All functions with the same derivative only differ in their constant terms.

Integral: antiderivative or primitive

Initial condition: the condition in which the solution to a differential equation must satisfy

Inscribed: drawn inside a given figure

Integrand: expression being integrated

Integration: the process of finding a definite or indefinite integral

Kinematics: the study of motion

Lower bound: a number for which each member of a set of numbers is less than or equal to that number

Mean (or average) value of a function: the quotient of the definite integral of $y = f(x)$ from a to b and the interval length

Method of exhaustion: a technique of evaluating areas by subdividing the region into as many smaller figures as are possible and whose areas can be readily calculated

Position function: a function representing the location of an object at time t

Primitive: antiderivative or integral

Upper bound: a number for which each member of a set of numbers is greater than or equal to that number

Velocity: rate of change of position

Suggested Answers

Section 1: Activity 1

1. $F(x) = x + C$

2. $\int x\,dx = \frac{x^2}{2} + C$

3. $\int x^2\,dx = \frac{x^3}{3} + C$

4. $F(x) = \frac{x^4}{4} + C$

5. a. $\int x^{-3}\,dx = \frac{x^{-3+1}}{-3+1} + C$

$= -\frac{x^{-2}}{2} + C$

b. $\int \sqrt{x}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$

$= \frac{2}{3}x^{\frac{3}{2}} + C$

c. $\int \frac{1}{x^4}\,dx = \int x^{-4}\,dx$

$= \frac{x^{-4+1}}{-4+1} + C$

$= -\frac{x^{-3}}{3} + C$

d. $\int \frac{1}{\sqrt[3]{x}}\,dx = \int x^{-\frac{1}{3}}\,dx$

$= \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C$

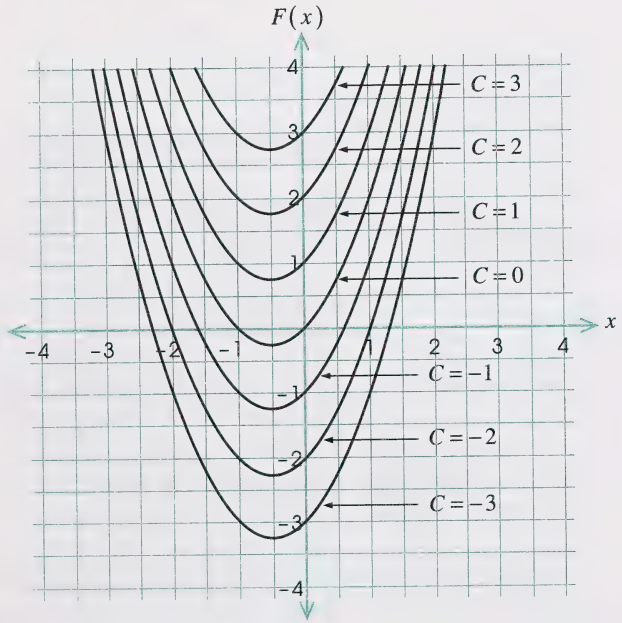
$= \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + C$

$= \frac{3}{2}x^{\frac{2}{3}} + C$

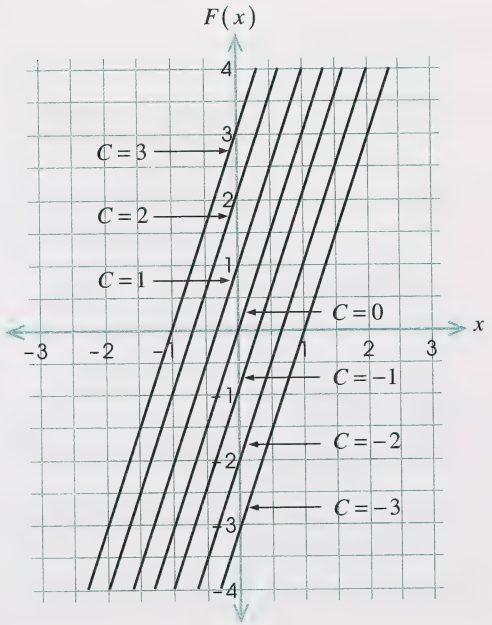
Section 1: Activity 2

1. Answers will vary. For these questions only a few integral values of C are used to demonstrate the families of curves.

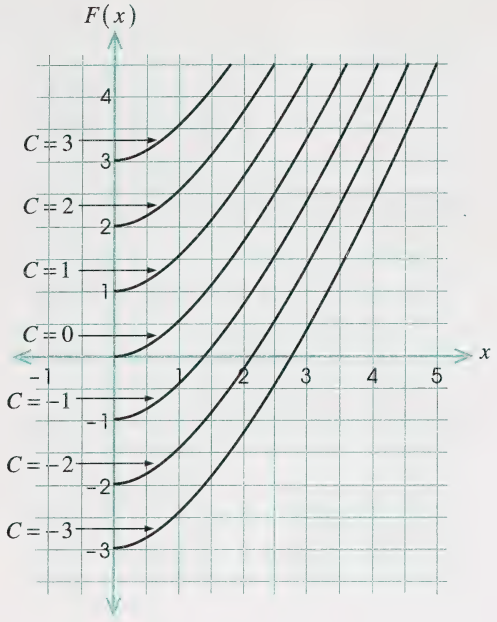
a. $F(x) = x^2 + x + C$



b. $F(x) = 3x + C$

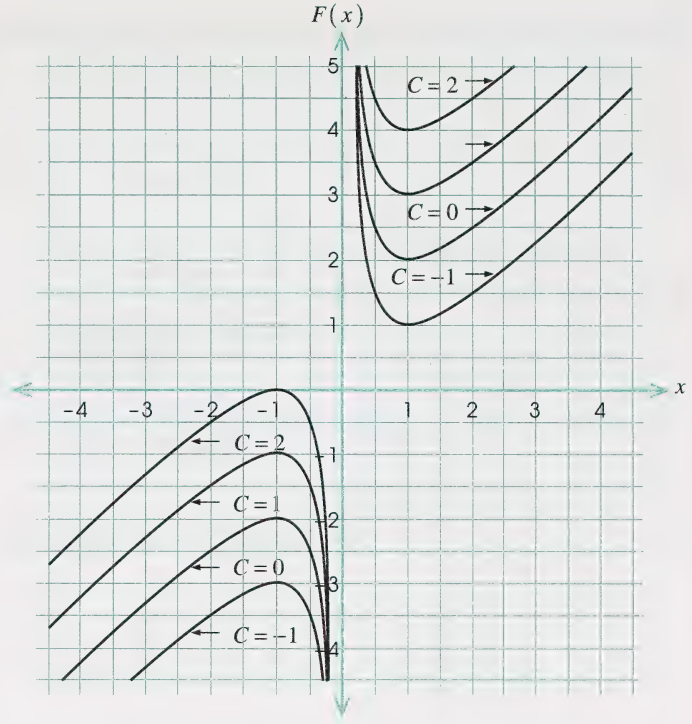


c. $F(x) = \frac{2}{3}x^{\frac{3}{2}} + C$



d. $F(x) = x - \frac{x^{-2+1}}{-2+1} + C$

$= x + x^{-1} + C$



2. a. Find the antiderivative.

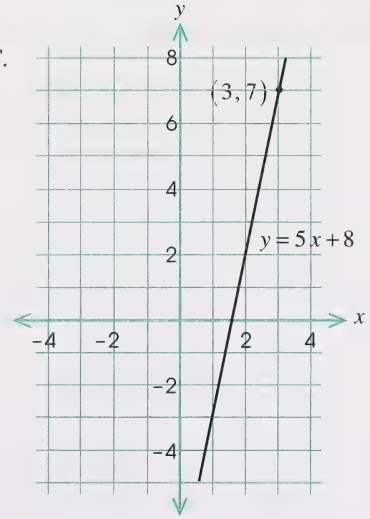
$y = \int 5 \, dx$

$= 5x + C$

Substitute $(3, 7)$ in the statement and evaluate C.

$$7 = 5(3) + C$$
$$C = -8$$

$$\therefore\ y = 5x - 8$$



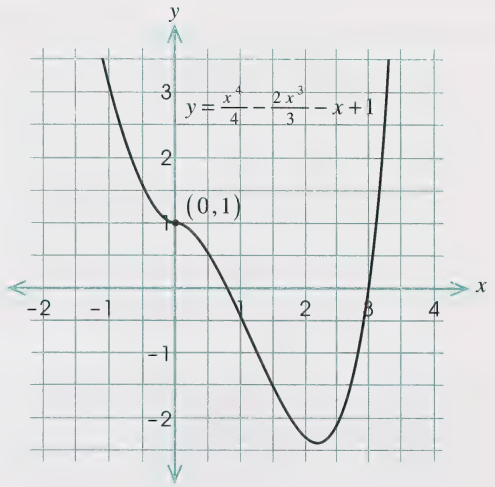
b. Find the antiderivative.

$$y = \int \left(x^3 - 2x^2 - 1 \right) dx$$
$$= \frac{x^4}{4} - \frac{2x^3}{3} - x + C$$

Substitute $(0, 1)$ in the statement and evaluate C.

$$1 = 0 - 0 - 0 + C$$
$$C = 1$$

$$\therefore\ y = \frac{x^4}{4} - \frac{2x^3}{3} - x + 1$$



Section 1: Activity 3

1. a. $\dfrac{dy}{dx} = 2x^{\frac{1}{3}} + 4x + 2$

$$y = \int \left(2x^{\frac{1}{3}} + 4x + 2 \right) dx$$
$$= \frac{2}{\frac{1}{3}+1} x^{\frac{1}{3}+1} + \frac{4}{1+1} x^{1+1} + 2x + C$$
$$= \frac{2}{\frac{4}{3}} x^{\frac{4}{3}} + \frac{4}{2} x^2 + 2x + C$$
$$= \frac{3}{2} x^{\frac{4}{3}} + 2x^2 + 2x + C$$

b. $\frac{dy}{dx} = (3x-1)^2$

$= 9x^2 - 6x + 1$

$y = \int (9x^2 - 6x + 1)\, dx$

$= \frac{9}{3}x^3 - \frac{6}{2}x^2 + x + C$

$= 3x^3 - 3x^2 + x + C$

c. $\frac{dv}{dt} = -10t + 5$

$v = \int (-10t + 5)\, dt$

$= -5t^2 + 5t + C$

2. a. $s = \int (2t + 4)\, dt$

$= t^2 + 4t + C$

Evaluate C by substituting the initial conditions.

$5 = 0^2 + 4(0) + C$

$C = 5$

$\therefore\ s = t^2 + 4t + 5$

b. $y = \int \frac{2}{x^2}\, dx$

$= \int (2x^{-2})\, dx$

$= -2x^{-1} + C$

Evaluate C by substituting the initial conditions.

$16 = -2(2^{-1}) + C$

$16 = -2\left(\frac{1}{2}\right) + C$

$16 = -1 + C$

$C = 17$

$\therefore\ y = -2x^{-1} + 17$

c. $\frac{dy}{dx} - 3x + 2 = 0$

$\frac{dy}{dx} = 3x - 2$

$y = \int (3x - 2)\, dx$

$= \frac{3}{2}x^2 - 2x + C$

Evaluate C by substituting the initial conditions.

$4 = \frac{3}{2}(1)^2 - 2(1) + C$

$4 = \frac{3}{2} - 2 + C$

$C = \frac{9}{2}$

$\therefore\ y = \frac{3}{2}x^2 - 2x + \frac{9}{2}$

Section 1: Activity 4

1. $\int \left(5x^2 - 3\right)^2 \, dx = \int \left(25x^4 - 30x^2 + 9\right) dx$

$$= \int 25x^4 \, dx - \int 30x^2 \, dx + \int 9 \, dx$$

$$= 25\int x^4 \, dx - 30\int x^2 \, dx + 9\int dx$$

$$= \frac{25}{5}x^5 - \frac{30}{3}x^3 + 9x + C$$

$$= 5x^5 - 10x^3 + 9x + C$$

2. $\int \left(x^{\frac{1}{3}} - 4x + 1\right) dx = \int x^{\frac{1}{3}} \, dx - \int 4x \, dx + \int 1 \, dx$

$$= \int x^{\frac{1}{3}} \, dx - 4\int x \, dx + 1\int dx$$

$$= \frac{x^{\frac{4}{3}}}{\frac{4}{3}} - \frac{4}{2}x^2 + x + C$$

$$= \frac{3x^{\frac{4}{3}}}{4} - 2x^2 + x + C$$

3. $\int \left(\frac{1}{3x^4} - 6x^4\right)^2 \, dx = \int \left(\frac{1}{9x^8} - 4 + 36x^8\right) dx$

$$= \int \frac{x^{-8}}{9} \, dx - \int 4 \, dx + \int 36x^8 \, dx$$

$$= \frac{x^{-7}}{9(-7)} - 4x + \frac{36x^9}{9} + C$$

$$= -\frac{1}{63x^7} - 4x + 4x^9 + C$$

4. $\int \left(\frac{2}{\sqrt[3]{x}} + \sqrt{x}\right) dx = \int \left(2x^{-\frac{1}{3}} + x^{\frac{1}{2}}\right) dx$

$$= \frac{2x^{\frac{2}{3}}}{\frac{2}{3}} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= 3x^{\frac{2}{3}} + \frac{2x^{\frac{3}{2}}}{3} + C$$

Section 1: Activity 5

1. a. If the question read $\int x^3\,dx$, the answer would be $\frac{x^4}{4}+C$.

Therefore, assume that the answer is of the form $y=\frac{a\left(x^2+1\right)^4}{4}$. If this function is the integral, then its derivative must equal $2x\left(x^2+1\right)^3$. The derivative of $y=\frac{a\left(x^2+1\right)^4}{4}+C$ can be obtained by applying the chain rule.

$$\frac{dy}{dx}=\frac{a(4)\left(x^2+1\right)^3(2x)}{4}$$
$$=2ax\left(x^2+1\right)^3$$

Compare this result with $2x\left(x^2+1\right)^3$, and solve for a.

$$2ax\left(x^2+1\right)^3=2x\left(x^2+1\right)^3$$
$$a=1$$

$$\therefore\ \int 2x\left(x^2+1\right)^3\,dx=\frac{\left(x^2+1\right)^4}{4}+C$$

b. If the question read $\int x^{\frac{1}{2}}\,dx$, the answer would be $\frac{2x^{\frac{3}{2}}}{3}+C$.

Therefore, assume that the answer is of the form $y=\frac{2a(1-x)^{\frac{3}{2}}}{3}$. If this function is the integral, then its derivative must equal $(1-x)^{\frac{1}{2}}$. The derivative of $y=\frac{2a(1-x)^{\frac{3}{2}}}{3}+C$ can be obtained by applying the chain rule.

$$\frac{dy}{dx}=\frac{2a(3)(1-x)^{\frac{1}{2}}(-1)}{2(3)}$$
$$=-a(1-x)^{\frac{1}{2}}$$

Compare this result with $(1-x)^{\frac{1}{2}}$, and solve for a.

$$-a(1-x)^{\frac{1}{2}}=(1-x)^{\frac{1}{2}}$$
$$a=-1$$

$$\therefore\ \int 2x\left(x^2+1\right)^3\,dx=\frac{-2(1-x)^{\frac{3}{2}}}{3}+C$$

c. If the question read $\int x^{-2}\,dx$, the answer would be $-x^{-1}+C$.

Therefore, assume that the answer is of the form $y=-a\left(x^3-1\right)^{-1}+C$. If this function is the integral, then its derivative must equal $6x^2\left(x^3-1\right)^{-2}$. The derivative of $y=-a\left(x^3-1\right)^{-1}+C$ can be obtained by applying the chain rule.

$$\frac{dy}{dx}=-a(-1)\left(x^3-1\right)^{-2}\left(3x^2\right)$$
$$=3ax^2\left(x^3-1\right)^{-2}$$

Compare this result with $6x^2\left(x^3-1\right)^{-2}$, and solve for a.

$$3ax^2\left(x^3-1\right)^{-2}=6x^2\left(x^3-1\right)^{-2}$$
$$a=2$$

$$\therefore\ \int 6x^2\left(x^3-1\right)^{-2}\,dx=-2\left(x^3-1\right)^{-1}+C$$

d. $\displaystyle\int\frac{3x}{\left(x^2+1\right)^3}\,dx=\int 3x\left(x^2+1\right)^{-3}\,dx$

If the question read $\int x^{-3}\,dx$, the answer would be $-\frac{x^{-2}}{2}+C$.

Therefore, assume that the answer is of the form $y=-\frac{a}{2}\left(x^2+1\right)^{-2}+C$. If this function is the integral, then its derivative must equal $3x\left(x^2+1\right)^{-3}$. The derivative of $y=-\frac{a}{2}\left(x^2+1\right)^{-2}+C$ can be obtained by applying the chain rule.

$$\frac{dy}{dx}=\frac{-a(-2)}{2}\left(x^2+1\right)^{-3}(2x)$$
$$=2ax\left(x^2+1\right)^{-3}$$

Compare this result with $3x\left(x^2+1\right)^{-3}$, and solve for a.

$$2ax\left(x^2+1\right)^{-3}=3x\left(x^2+1\right)^{-3}$$
$$a=\frac{3}{2}$$

$$\therefore\ \int\frac{3x}{\left(x^2+1\right)^3}\,dx=-\frac{3}{4}\left(x^2+1\right)^{-2}+C$$

2. $\int (x-1)^2 \, dx$

Method 1: Using Comparison Techniques

Assume that the solution is of the form $y = \frac{(x-1)^3}{3}$.

Check the answer by differentiating.

$$\frac{dy}{dx} = \frac{3(x-1)^2 (1)}{3}$$
$$= (x-1)^2$$

$$\therefore \int (x-1)^2 \, dx = \frac{(x-1)^3}{3} + C$$

Method 2: Expanding and Integrating

$$\int (x-1)^2 \, dx = \int \left(x^2 - 2x + 1\right) dx$$
$$= \frac{x^3}{3} - x^2 + x + C$$

Are these two answers equivalent?

Begin by expanding the answer from Method 1.

$$\frac{(x-1)^3}{3} + C = \frac{x^3 - 3x^2 + 3x - 1}{3} + C$$
$$= \frac{x^3}{3} - x^2 + x - \frac{1}{3} + C$$

Now, since C is arbitrary, $-\frac{1}{3} + C$ is also arbitrary and could be represented by C_1.

Therefore, the right side of the preceding expansion can be written $\frac{x^3}{3} - x^2 + x + C_1$.

This is exactly the answer from Method 2!

Section 1: Activity 6

1. Because $\int \sin x \, dx = -\cos x$, assume that the primitive of $f(x) = \sin (3x-1)$ is of the form $F(x) = -a \cos (3x-1) + C$.

$$\frac{d}{dx}[F(x)] = -a[-\sin (3x-1)](3)$$
$$= 3a \sin (3x-1)$$

$$\therefore\ 3a\sin(3x-1) = \sin(3x-1)$$
$$3a = 1$$
$$a = \frac{1}{3}$$

$$\therefore\ F(x) = -\frac{1}{3}\cos(3x-1) + C$$

2. Since $\int \sec x \tan x\,dx = \sec x + C$, assume the solution to $\int 3\sec 2x \tan 2x\,dx$ is of the form $y = a\sec 2x + C$.

$$\frac{dy}{dx} = a(\sec 2x \tan 2x)(2)$$
$$= 2a\sec 2x \tan 2x$$

$$\therefore\ 2a\sec 2x \tan 2x = 3\sec 2x \tan 2x$$
$$2a = 3$$
$$a = \frac{3}{2}$$

$$\therefore\ \int 3\sec 2x \tan 2x\,dx = \frac{3}{2}\sec 2x + C$$

3. $\dfrac{dy}{dx} = \csc^2 3x\,dx$

Since $\int \csc^2 x\,dx = -\cot x + C$, assume that the solution to the differential equation is $y = -a\cot 3x + C$.

$$\frac{dy}{dx} = -a\left[-\csc^2(3x)\right](3)$$
$$= 3a\csc^2(3x)$$

$$\therefore\ 3a\csc^2(3x) = \csc^2(3x)$$
$$3a = 1$$
$$a = \frac{1}{3}$$

$$\therefore\ y = -\frac{1}{3}\cot 3x + C$$

4. $y' = 2\sin x \cos x$

Method 1: In Terms of sin *x*

Assume $y = \frac{a\sin^2 x}{2} + C$.

$$\frac{dy}{dx} = \frac{a(2\sin x)(\cos x)}{2}$$
$$= a\sin x \cos x$$

$$\therefore\ a \sin x \cos x = 2 \sin x \cos x$$
$$a = 2$$

$$\therefore\ y = \frac{2 \sin^2 x}{2} + C$$
$$= \sin^2 x + C$$

Method 2: In Terms of cos *x*

Assume $y = \frac{a \cos^2 x}{2} + C$.

$$\frac{dy}{dx} = \frac{a(2 \cos x)(-\sin x)}{2}$$
$$= -a \sin x \cos x$$

$$\therefore\ -a \sin x \cos x = 2 \sin x \cos x$$
$$a = -2$$

$$\therefore\ y = \frac{-2 \cos^2 x}{2} + C$$
$$= -\cos^2 x + C$$

Next, show that the answers are equivalent.

$$\sin^2 x + C = \sin^2 x + C_1 - 1, \text{ where } C = C_1 - 1$$
$$= (\sin^2 x - 1) + C_1$$
$$= -\cos^2 x + C_1$$

since $\sin^2 x + \cos^2 x = 1$
or $\sin^2 x - 1 = -\cos^2 x$

Since C is arbitrary, the answers are equivalent.

5. $\int 5 \sin^3 (4x+1) \cos (4x+1)\, dx$

Assume that the answer is of the form $y = \frac{a \sin^4 (4x+1)}{4}$.

$$\frac{dy}{dx} = \frac{a[4 \sin^3 (4x+1)][\cos (4x+1)](4)}{4}$$
$$= 4a \sin^3 (4x+1) \cos (4x+1)$$

$$4a \sin^3 (4x+1) \cos (4x+1) = 5 \sin^3 (4x+1) \cos (4x+1)$$
$$4a = 5$$
$$a = \frac{5}{4}$$

$$\therefore\ \int 5 \sin^3 (4x+1) \cos (4x+1)\, dx = \frac{5 \sin^4 (4x+1)}{16} + C$$

6. Solve $\frac{dy}{dx} = x^2 \cos\left(x^3 + 1\right)$ if $y = 1$ when $x = -1$.

Assume that the solution is of the form $y = a \sin\left(x^3 + 1\right) + C$.

$$\frac{dy}{dx} = a\left[\cos\left(x^3 + 1\right)\right]\left(3x^2\right)$$
$$= 3ax^2 \cos\left(x^3 + 1\right)$$

$$\therefore\ 3ax^2 \cos\left(x^3 + 1\right) = x^2 \cos\left(x^3 + 1\right)$$
$$3a = 1$$
$$a = \frac{1}{3}$$

$$\therefore\ y = \frac{1}{3}\sin\left(x^3 + 1\right) + C$$

Now find C by substituting $x = -1$ and $y = 1$.

$$1 = \frac{1}{3}\sin\left[(-1)3 + 1\right] + C$$
$$1 = \frac{1}{3}\sin 0 + C$$
$$1 = 0 + C$$
$$C = 1$$

$$\therefore\ y = \frac{1}{3}\sin\left(x^3 + 1\right) + 1$$

7. $\frac{dy}{dx} = \frac{\sin x}{\cos^3 x}$

$$= \cos^{-3} x \sin x$$

Assume $y = \frac{a\cos^{-2} x}{-2} + C$.

$$\frac{dy}{dx} = \frac{a}{-2}\left(-2\cos^{-3} x\right)(-\sin x)$$
$$= -a\cos^{-3} x \sin x$$

$$\therefore\ -a\cos^{-3} x \sin x = \cos^{-3} x \sin x$$
$$a = -1$$

$$\therefore\ y = \frac{1}{2}\cos^{-2} x + C$$

Section 1: Follow-up Activities

Extra Help

1. Recall $\int a\cos bx\,dx = \frac{a}{b}\sin bx + C$. In this question, $a = 6$ and $b = 7$.

$$\therefore\ \int 6\cos 7x\,dx = \frac{6}{7}\sin 7x + C$$

2. Recall $\int a \sin bx\, dx = -\frac{a}{b}\cos bx + C$. Here, $a = 2$ and $b = -3$.

$$\int 2 \sin(-3x)\, dx = \frac{-2}{-3}\cos(-3x) + C$$
$$= \frac{2}{3}\cos(-3x) + C$$

3. Recall $\int a \cos bx\, dx = \frac{a}{b}\sin bx + C$. In this question, $a = 5$ and $b = \frac{1}{5}$.

$$\int 5 \cos \frac{x}{5}\, dx = \frac{5}{\frac{1}{5}}\sin 5x + C$$
$$= 25 \sin 5x + C$$

4. Recall $\int a \sin bx\, dx = -\frac{a}{b}\cos bx + C$. Here, $a = \frac{1}{3}$ and $b = -\frac{1}{3}$.

$$\int \frac{1}{3}\sin\left(-\frac{x}{3}\right) dx = \frac{\frac{1}{3}}{-\frac{1}{3}}\cos\left(-\frac{x}{3}\right) + C$$
$$= -\cos\left(-\frac{x}{3}\right) + C$$

Enrichment

1. Let $f(x) = \sin x$ and $g'(x) = x$.

Therefore, $f'(x) = \cos x$ and $g(x) = \frac{x^2}{2}$.

Substitute into

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx + C.$$

$$\therefore \int x \sin x\, dx = \frac{x^2}{2}\sin x - \int \frac{x^2}{2}\cos x\, dx + C$$

Notice that the integral on the right side is no easier to evaluate than the original problem.

2. Let $f(x) = x$ and $g'(x) = \cos x$.

Therefore, $f'(x) = 1$ and $g(x) = \sin x$.

Substitute into

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx + C.$$

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx + C$$
$$= x \sin x - (-\cos x) + C$$
$$= x \sin x + \cos x + C$$

3. Let $f(x) = 2x + 1$ and $g'(x) = \sin 2x$.

Therefore, $f'(x) = 2$ and $g(x) = -\frac{1}{2}\cos 2x$.

Substitute into

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx + C.$$

$$\begin{aligned}
&\int (2x+1)\sin 2x\,dx \\
&\quad = -\frac{1}{2}(2x+1)\cos 2x - \int\left(-\frac{1}{2}\cos 2x\right)(2)\,dx + C \\
&\quad = -\frac{1}{2}(2x+1)\cos 2x + \int \cos 2x\,dx + C \\
&\quad = -\frac{1}{2}(2x+1)\cos 2x + \frac{1}{2}\sin 2x + C
\end{aligned}$$

Section 2: Activity 1

1. a. $y = \frac{1}{2}x$, where $\Delta x = 1$

x	0	1	2	3	4	5
y	0	0.5	1	1.5	2	2.5

The calculations for both the inscribed rectangles and the circumscribed rectangles are as follows:

Inscribed Rectangles		
Δx	Height	Area
1	0.0	0.0
1	0.5	0.5
1	1.0	1.0
1	1.5	1.5
1	2.0	2.0
Total Area		5.0

Circumscribed Rectangles		
Δx	Height	Area
1	0.5	0.5
1	1.0	1.0
1	1.5	1.5
1	2.0	2.0
1	2.5	2.5
Total Area		7.5

b. $\Delta x = 0.5$

x	0	0.5	1	1.5	2	2.5
y	0	0.25	0.5	0.75	1	1.25

x	3	3.5	4	4.5	5
y	1.5	1.75	2	2.25	2.5

The calculations for both the inscribed rectangles and the circumscribed rectangles are as follows:

Inscribed Rectangles		
Δx	Height	Area
0.5	0.00	0.000
0.5	0.25	0.125
0.5	0.50	0.250
0.5	0.75	0.375
0.5	1.00	0.500
0.5	1.25	0.625
0.5	1.50	0.750
0.5	1.75	0.875
0.5	2.00	1.000
0.5	2.25	1.125
Total Area		5.625

Circumscribed Rectangles		
Δx	Height	Area
0.5	0.25	0.125
0.5	0.50	0.250
0.5	0.75	0.375
0.5	1.00	0.500
0.5	1.25	0.625
0.5	1.50	0.750
0.5	1.75	0.875
0.5	2.00	1.000
0.5	2.25	1.125
0.5	2.50	1.250
Total Area		6.875

c. The true area is $\frac{1}{2}(6.875+5.625)=6.25 \text{ units}^2$.

The average area is not always the true area.

Since the graph is a straight line, you can also use the area formula for a triangle $\left(A=\frac{1}{2}bh\right)$ to find the true area. For example, $\frac{1}{2}(5)(2.5)=6.25 \text{ units}^2$.

2. a. $y^2=x-3$

$y=\sqrt{x-3}$

x	3	4	5	6	7	8
y	0	1	1.414	1.732	2	2.236

The calculations for both the inscribed rectangles and the circumscribed rectangles are as follows:

Inscribed Rectangles		
Δx	Height	Area
1	0.000	0.000
1	1.000	1.000
1	1.414	1.414
1	1.732	1.732
1	2.000	2.000
Total Area		6.146

Circumscribed Rectangles		
Δx	Height	Area
1	1.000	1.000
1	1.414	1.414
1	1.732	1.732
1	2.000	2.000
1	2.236	2.236
Total Area		8.382

b. $\Delta x=0.5$

x	3	3.5	4	4.5	5	5.5
y	0	0.707	1	1.225	1.414	1.581

x	6	6.5	7	7.5	8
y	1.732	1.871	2	2.121	2.236

The calculations for both the inscribed rectangles and the circumscribed rectangles are as follows:

Inscribed Rectangles		
Δx	Height	Area
0.5	0.000	0.000
0.5	0.707	0.354
0.5	1.000	0.500
0.5	1.225	0.613
0.5	1.414	0.707
0.5	1.581	0.791
0.5	1.732	0.866
0.5	1.871	0.936
0.5	2.000	1.000
0.5	2.121	1.061
Total Area		6.828

Circumscribed Rectangles		
Δx	Height	Area
0.5	0.707	0.354
0.5	1.000	0.500
0.5	1.225	0.613
0.5	1.414	0.707
0.5	1.581	0.791
0.5	1.732	0.866
0.5	1.871	0.936
0.5	2.000	1.000
0.5	2.121	1.061
0.5	2.236	1.118
Total Area		7.946

c. $\text{Average Area} = \dfrac{7.946 + 6.828}{2}$

$= 7.387$

Thus, the average area is 7.387 units^2.

3. $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

For $n = 1$, verify that $1 = \frac{1(1+1)}{2}$.

LS	RS
1	$\dfrac{1(1+1)}{2}$
	$= \dfrac{1(2)}{2}$
	$= 1$

LS = RS

For $n = 2$, verify that $1 + 2 = \frac{2(2+1)}{2}$.

LS	RS
$1 + 2$	$\dfrac{2(2+1)}{2}$
$= 3$	$= \dfrac{2(3)}{2}$
	$= 3$

LS = RS

For $n = 3$, verify that $1+2+3 = \frac{3(3+1)}{2}$.

LS	RS
$1+2+3$	$\frac{3(3+1)}{2}$
$=6$	$=\frac{3(4)}{2}$
	$=6$

LS = RS

For $n = 4$, verify that $1+2+3+4 = \frac{4(4+1)}{2}$.

LS	RS
$1+2+3+4$	$\frac{4(4+1)}{2}$
$=10$	$=\frac{4(5)}{2}$
	$=10$

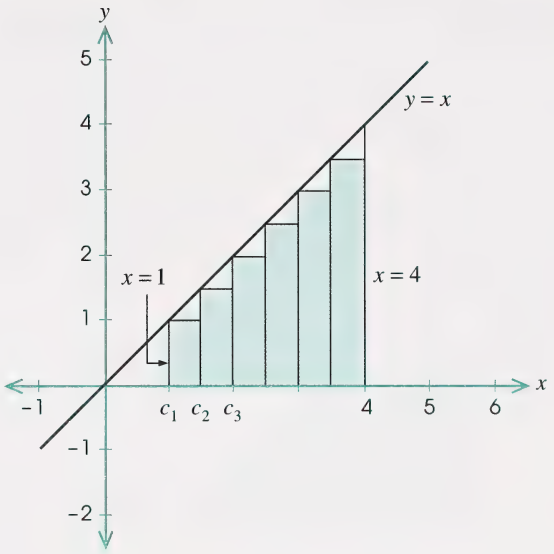
LS = RS

For $n = 5$, verify that $1+2+3+4+5 = \frac{5(5+1)}{2}$.

LS	RS
$1+2+3+4+5$	$\frac{5(5+1)}{2}$
$=15$	$=\frac{5(6)}{2}$
	$=15$

LS = RS

4.



Divide the interval $[1, 4]$ into n strips. Because that interval is 3 units wide, each of the n-strips will be of width $\Delta x = \frac{3}{n}$. The x-values, $c_1, c_2, c_3, \ldots, c_n$, at which the lengths $f(c_1), f(c_2), f(c_3), \ldots, f(c_n)$ of the inscribed rectangles occur, can be expressed in terms of Δx.

$$c_1 = 1 \qquad c_2 = c_1 + \Delta x = 1 + \Delta x \qquad c_3 = c_2 + \Delta x = (1 + \Delta x) + \Delta x = 1 + 2\Delta x$$

Therefore, $c_n = 1 + (n-1)\Delta x$.

The areas of the inscribed rectangles are as follows:

$$f(c_1)\Delta x = (1)\Delta x = 1\Delta x$$

$$f(c_2)\Delta x = (1 + \Delta x)\Delta x$$

$$f(c_3)\Delta x = (1 + 2\Delta x)\Delta x$$

$$\therefore f(c_n)\Delta x = [1 + (n-1)\Delta x]\Delta x$$

Next, find the sum of the n rectangles.

$$\begin{aligned}
\text{sum} &= \Delta x + (1 + \Delta x)\Delta x + (1 + 2\Delta x)\Delta x + (1 + 3\Delta x)\Delta x \\
&\quad + \ldots + [1 + (n-1)\Delta x]\Delta x \\
&= \Delta x + [\Delta x + (\Delta x)^2] + [\Delta x + 2(\Delta x)^2] \\
&\quad + [\Delta x + 3(\Delta x)^2] + \ldots + [\Delta x + (n-1)(\Delta x)^2] \\
&= \underbrace{(\Delta x + \Delta x + \Delta x + \ldots + \Delta x)}_{n \text{ terms}} + [1 + 2 + 3 + \ldots + (n-1)^2](\Delta x)^2 \\
&= n\Delta x + \theta - [1 + 2 + 3 + \ldots + (n-1)^2](\Delta x)^2
\end{aligned}$$

Now, replace Δx by $\frac{3}{n}$. Recall from Mathematics 30, that the formula for the sum of the n naturals is $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

$$\therefore\ 1 + 2 + 3 + \ldots + (n-1) = \frac{(n-1)(n)}{2}$$

Substitute.

$$\begin{aligned}
\text{sum} &= n\left(\frac{3}{n}\right) + \frac{(n-1)(n)}{2} \bullet \left(\frac{9}{n^2}\right) \\
&= 3 + \frac{9(n-1)}{2n} \\
&= 3 + \frac{9n}{2n} - \frac{9}{2n} \\
&= \frac{15}{2} - \frac{9}{2n}
\end{aligned}$$

Now, the exact area will be the limit of this sum as the number of strips increases without bound.

$A = \lim_{n\to\infty}$ (sum of n rectangles)

or

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \bullet \Delta x$$

$$= \lim_{n\to\infty} \left(\frac{15}{2} - \frac{9}{2n} \right)$$

$$= \frac{15}{2}$$

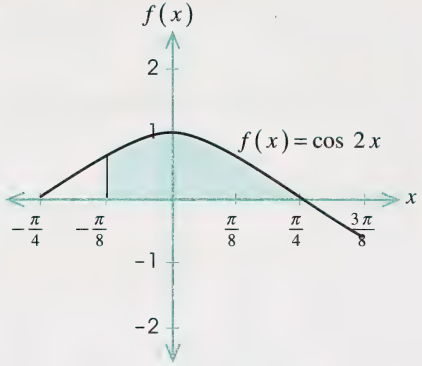
This answer may be checked by using the formula for a trapezoid $A = \frac{h}{2}(b_1 + b_2)$, where $h =$ distance between the parallel sides b_1 and b_2.

$$A = \frac{3}{2}(1+4)$$

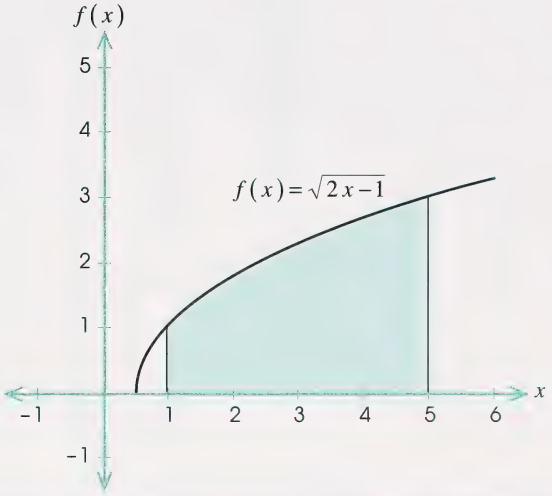
$$= \frac{15}{2}$$

Section 2: Activity 2

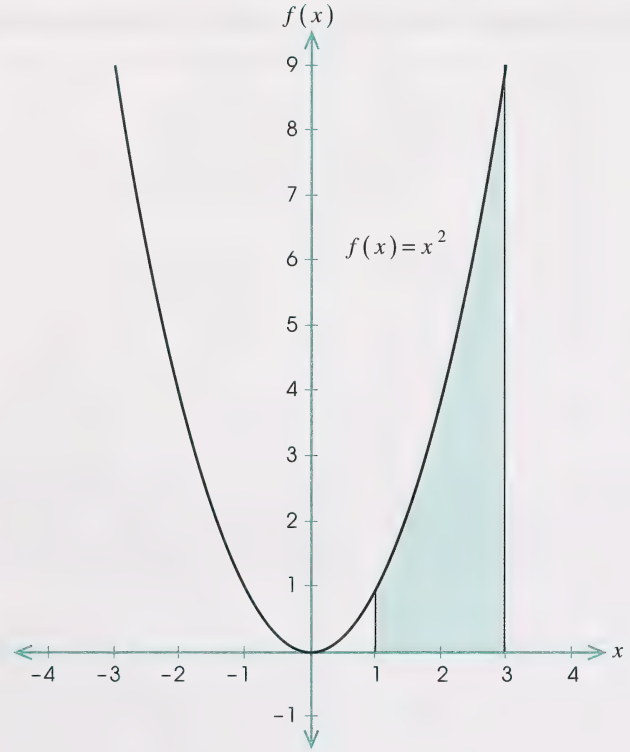
1. a.



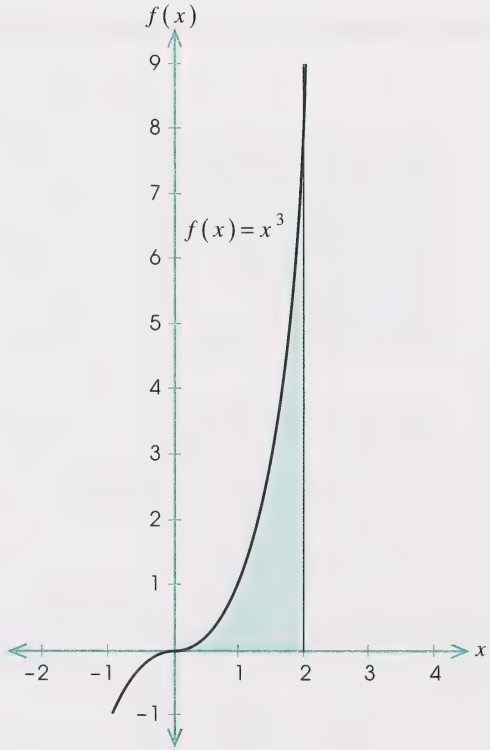
b.



c.



d.



2. a. $f(x) = \cos 2x$

$$F(x) = \frac{\sin 2x}{2}$$

$$_{-\frac{\pi}{8}}A_{\frac{\pi}{4}} = \int_{-\frac{\pi}{8}}^{\frac{\pi}{4}} \cos 2x \, dx$$

$$= \left[\frac{\sin 2x}{2}\right]_{-\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= \frac{\sin 2\left(\frac{\pi}{4}\right)}{2} - \frac{\sin 2\left(-\frac{\pi}{8}\right)}{2}$$

$$= \frac{\sin\left(\frac{\pi}{2}\right)}{2} - \frac{\sin\left(-\frac{\pi}{4}\right)}{2}$$

$$= \frac{1}{2} - \frac{\left(-\frac{\sqrt{2}}{2}\right)}{2}$$

$$= \frac{1}{2} + \frac{\sqrt{2}}{4}$$

$$= \frac{2+\sqrt{2}}{4}$$

b. $f(x) = \sqrt{2x-1}$

$$F(x) = \frac{(2x-1)^{\frac{3}{2}}}{3}$$

Check the integral by differentiating.

$$\frac{d}{dx}F(x) = \frac{3}{2} \cdot \frac{(2x-1)^{\frac{3}{2}-1}(2)}{3}$$

$$= (2x-1)^{\frac{1}{2}}$$

$$_1A_5 = \int_1^5 \sqrt{2x-1} \, dx$$

$$= \left[\frac{(2x-1)^{\frac{3}{2}}}{3}\right]_1^5$$

$$= \frac{(2\cdot 5-1)^{\frac{3}{2}}}{3} - \frac{(2\cdot 1-1)^{\frac{3}{2}}}{3}$$

$$= \frac{9^{\frac{3}{2}}}{3} - \frac{1^{\frac{3}{2}}}{3}$$

$$= \frac{\left(\sqrt{9}\right)^3}{3} - \frac{1}{3}$$

$$= 9 - \frac{1}{3}$$

$$= \frac{26}{3}$$

c. $f(x) = x^2$

$F(x) = \frac{x^3}{3}$

$$
\begin{aligned}
{}_1A_3 &= \int_1^3 x^2\,dx \\
&= \left[\frac{x^3}{3}\right]_1^3 \\
&= \frac{3^3}{3} - \frac{1}{3} \\
&= \frac{27}{3} - \frac{1}{3} \\
&= \frac{26}{3}
\end{aligned}
$$

d. $f(x) = x^3$

$F(x) = \frac{x^4}{4}$

$$
\begin{aligned}
\int_0^2 x^3\,dx &= \left[\frac{x^4}{4}\right]_0^2 \\
&= \frac{2^4}{4} - \frac{0^4}{4} \\
&= \frac{16}{4} - 0 \\
&= 4
\end{aligned}
$$

Section 2: Activity 3

1.

LS	RS
$\int_1^2 \left(2x + 3x^2\right)\,dx$	$\int_1^2 2x\,dx + \int_1^2 3x^2\,dx$
$= \left[x^2 + x^3\right]_1^2$	$= \left[x^2\right]_1^2 + \left[x^3\right]_1^2$
$= \left(2^2 + 2^3\right) - \left(1^2 + 1^3\right)$	$= \left(2^2 - 1^2\right) + \left(2^3 - 1^3\right)$
$= (4 + 8) - (1 + 1)$	$= (4 - 1) + (8 - 1)$
$= 12 - 2$	$= 3 + 7$
$= 10$	$= 10$

LS = RS

2.

LS	RS
$\int_{-1}^{2}(3+2x)\,dx$	$-\int_{2}^{-1}(3+2x)\,dx$
$=\left[3x+x^2\right]_{-1}^{2}$	$=-\left[3x+x^2\right]_{2}^{-1}$
$=\left[3(2)+2^2\right]-\left[3(-1)+(-1)^2\right]$	$=-\left\{\left[3(-1)+(-1)^2\right]-\left[3(2)+2^2\right]\right\}$
$=(6+4)-(-3+1)$	$=-\left[(-3+1)-(6+4)\right]$
$=10-(-2)$	$=-(-2-10)$
$=12$	$=-(-12)$
	$=12$

LS = RS

3. a. $\int_{-3}^{2}\left(3x^2-4\right)dx=\left[x^3-4x\right]_{-3}^{2}$

$=\left[(2)^3-4(2)\right]-\left[(-3)^3-4(-3)\right]$

$=(8-8)-(-27+12)$

$=15$

b. $\int_{-1}^{3}\left[\frac{1}{(3x-2)^2}\right]dx=\int_{-1}^{3}(3x-2)^{-2}\,dx$

Assume the primitive is of the form $F(x)=-a(3x-2)^{-1}$.

$$F'(x) = -a(-1)(3x-2)^{-2}(3)$$
$$= 3a(3x-2)^{-2}$$

Equate to the integrand, and solve for a.

$$3a(3x-2)^{-2} = (3x-2)^{-2}$$
$$3a = 1$$
$$a = \frac{1}{3}$$

$$\therefore F(x) = -\frac{1}{3}(3x-2)^{-1}$$

$$\int_{-1}^{3}\left[\frac{1}{(3x-2)^2}\right]dx = \left[-\frac{1}{3}(3x-2)^{-1}\right]_{-1}^{3}$$
$$= -\frac{1}{3[3(3)-2]} + \frac{1}{3[3(-1)-2]}$$
$$= -\frac{1}{21} + \frac{1}{-15}$$
$$= -\frac{12}{105}$$
$$= -\frac{4}{35}$$

c. $\int_3^4 \left(2x^2+3\right)^2\,dx = \int_3^4 \left(4x^4+12x^2+9\right)dx$

$$= \left[\frac{4}{5}x^5 + 4x^3 + 9x\right]_3^4$$
$$= \frac{4}{5}(4)^5 + 4(4)^3 + 9(4) - \frac{4}{5}(3)^5$$
$$-4(3)^3 - 9(3)$$
$$= \frac{4096}{5} + 256 + 36 - \frac{972}{5} - 108 - 27$$
$$= \frac{3124}{5} + 157$$
$$= 781\frac{4}{5}$$

d. Assume the primitive is of the form $F(x) = \frac{a(x-3)^{\frac{3}{2}}}{\frac{3}{2}}$.

$$\frac{d}{dx}F(x) = \frac{2a}{3}\left(\frac{3}{2}\right)(x-3)^{\frac{1}{2}}$$
$$= a(x-3)^{\frac{1}{2}}$$

Equate this result to the integrand.

$$a(x-3)^{\frac{1}{2}} = (x-3)^{\frac{1}{2}}$$
$$a = 1$$

$\therefore F(x) = \frac{2}{3}(x-3)^{\frac{3}{2}}$

$$\int_7^{12} (x-3)^{\frac{1}{2}}\, dx = \left[\frac{2}{3}(x-3)^{\frac{3}{2}}\right]_7^{12}$$
$$= \frac{2}{3}(12-3)^{\frac{3}{2}} - \frac{2}{3}(7-3)^{\frac{3}{2}}$$
$$= \frac{2}{3}(9)^{\frac{3}{2}} - \frac{2}{3}(4)^{\frac{3}{2}}$$
$$= \frac{2}{3}(27) - \frac{2}{3}(8)$$
$$= 18 - \frac{16}{3}$$
$$= \frac{38}{3} \text{ or } 12\frac{2}{3}$$

e. Assume the primitive is of the following form:

$$F(x) = \frac{a\left(x^2-1\right)^{\frac{3}{2}}}{\frac{3}{2}}$$
$$= \frac{2}{3}a\left(x^2-1\right)^{\frac{3}{2}}$$

$$F'(x) = \frac{2}{3}a\left(\frac{3}{2}\right)\left(x^2-1\right)^{\frac{1}{2}}(2x)$$
$$= 2ax\left(x^2-1\right)^{\frac{1}{2}}$$

Solve for a.

$$2ax\left(x^2-1\right)^{\frac{1}{2}} = 2x\left(x^2-1\right)^{\frac{1}{2}}$$
$$a = 1$$

$$\therefore F(x) = \frac{2}{3}\left(x^2-1\right)^{\frac{3}{2}}$$

$$\int_{-2}^{1} 2x\left(x^2-1\right)^{\frac{1}{2}}\, dx = \left[\frac{2}{3}\left(x^2-1\right)^{\frac{3}{2}}\right]_{-2}^{1}$$
$$= \frac{2}{3}\left[(1-1)^{\frac{3}{2}} - (4-1)^{\frac{3}{2}}\right]$$
$$= \frac{2}{3}\left(0 - \sqrt{3^3}\right)$$
$$= \frac{2}{3}\left(0 - 3\sqrt{3}\right)$$
$$= -2\sqrt{3}$$

f. Assume the primitive is of the following form:

$$F(x) = \frac{a(5-x)^{\frac{3}{2}}}{\frac{3}{2}}$$
$$= \frac{2a}{3}(5-x)^{\frac{3}{2}}$$

$$F'(x) = \frac{2}{3}a\left(\frac{3}{2}\right)(5-x)^{\frac{1}{2}}(-1)$$
$$= -a(5-x)^{\frac{1}{2}}$$

Solve for a.

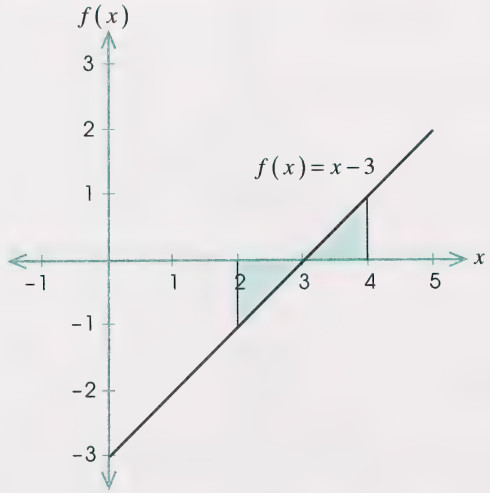
$$-a(5-x)^{\frac{1}{2}} = (5-x)^{\frac{1}{2}}$$
$$-a = 1$$
$$a = -1$$

$$\therefore F(x) = -\frac{2}{3}(5-x)^{\frac{3}{2}}$$

$$\int_{-1}^{2}(5-x)^{\frac{1}{2}}dx = \left[-\frac{2}{3}(5-x)^{\frac{3}{2}}\right]_{-1}^{2}$$
$$= \frac{-2(5-2)^{\frac{3}{2}}}{3} + \frac{2(5+1)^{\frac{3}{2}}}{3}$$
$$= \frac{-2\left(3\sqrt{3}\right)}{3} + \frac{2\left(6\sqrt{6}\right)}{3}$$
$$= -2\sqrt{3} + 4\sqrt{6}$$

Section 2: Activity 4

1. a.

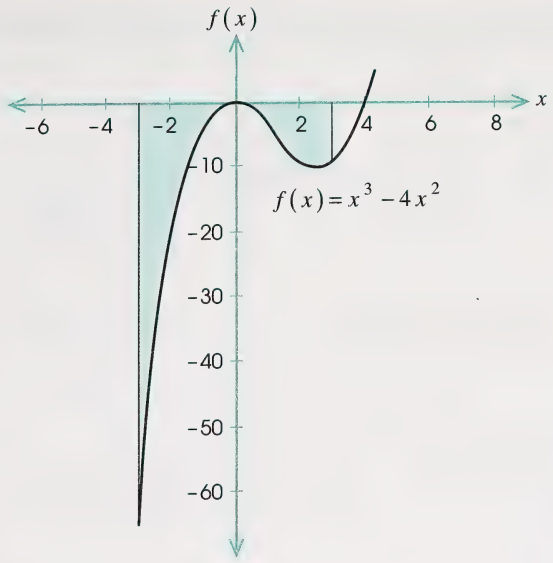


The x-intercept is 3.

$$_2A_4 = \left|\int_2^3 (x-3)\,dx\right| + \left|\int_3^4 (x-3)\,dx\right|$$
$$= \left|\left[\frac{x^2}{2} - 3x\right]_2^3\right| + \left|\left[\frac{x^2}{2} - 3x\right]_3^4\right|$$
$$= |4.5 - 9 - 2 + 6| + |8 - 12 - 4.5 + 9|$$
$$= 0.5 + 0.5$$
$$= 1$$

The area is 1.

b.



Let $x^3 - 4x^2 = 0$.

$x^2(x-4) = 0$

$x^2 = 0$ or $x - 4 = 0$

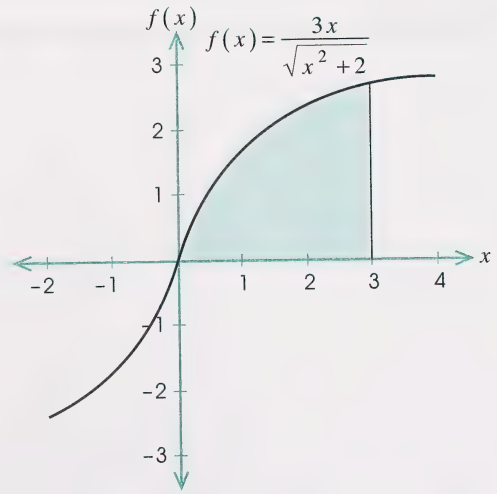
$x = 0$ $\qquad$ $x = 4$

$$
\begin{aligned}
{}_{-3}A_3 &= {}_{-3}A_0 + {}_0A_3 \\
&= \left|\int_{-3}^{0}\left(x^3 - 4x^2\right)dx\right| + \left|\int_{0}^{3}\left(x^3 - 4x^2\right)dx\right| \\
&= \left|\left[\frac{x^4}{4} - \frac{4}{3}x^3\right]_{-3}^{0}\right| + \left|\left[\frac{x^4}{4} - \frac{4x^3}{3}\right]_{0}^{3}\right| \\
&= \left|0 - \frac{(-3)^4}{4} + \frac{4}{3}(-3)^3\right| + \left|\frac{3^4}{4} - \frac{4(3)^3}{3} - 0\right| \\
&= \left|\frac{-81}{4} - \frac{108}{3}\right| + \left|\frac{81}{4} - \frac{108}{3}\right| \\
&= \left|\frac{-243-432}{12}\right| + \left|\frac{243-432}{12}\right| \\
&= \left|-\frac{675}{12}\right| + \left|\frac{-189}{12}\right| \\
&= \frac{675}{12} + \frac{189}{12} \\
&= 72
\end{aligned}
$$

The area is 72.

Note: The area could also have been found by using $\int_{-3}^{3}\left(x^3 - 4x^2\right)dx = \left[\frac{x^4}{4} - \frac{4}{3}x^3\right]_{-3}^{3}$ since the curve does not cross the x-axis in the required interval.

c.



$$
\begin{aligned}
{}_0A_3 &= \int_0^3 \left(\frac{3x}{\sqrt{x^2+2}} \right) dx \\
&= \int_0^3 3x\left(x^2+2\right)^{-\frac{1}{2}}\, dx
\end{aligned}
$$

Assume the primitive is of the following form:

$$
\begin{aligned}
F(x) &= \frac{a\left(x^2+2\right)^{\frac{1}{2}}}{\frac{1}{2}} \\
&= 2a\left(x^2+2\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
F'(x) &= 2a\left(\frac{1}{2}\right)\left(x^2+2\right)^{-\frac{1}{2}}(2x) \\
&= 2ax\left(x^2+2\right)^{-\frac{1}{2}}
\end{aligned}
$$

Solve for a.

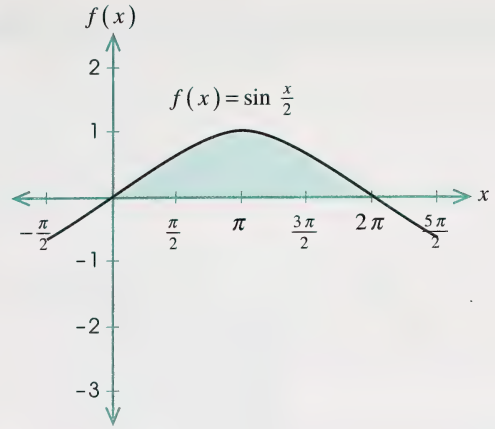
$$
\begin{aligned}
2ax\left(x^2+2\right)^{-\frac{1}{2}} &= 3x\left(x^2+2\right)^{-\frac{1}{2}} \\
2a &= 3 \\
a &= \frac{3}{2}
\end{aligned}
$$

$$\therefore\ F(x) = 3\left(x^2+2\right)^{\frac{1}{2}}$$

$$
\begin{aligned}
\int_0^3 \left(\frac{3x}{\sqrt{x^2+2}} \right) dx &= \left[3\sqrt{x^2+2} \right]_0^3 \\
&= 3\left(\sqrt{9+2} - \sqrt{2}\right) \\
&= 3\sqrt{11} - 3\sqrt{2} \\
&\doteq 5.7
\end{aligned}
$$

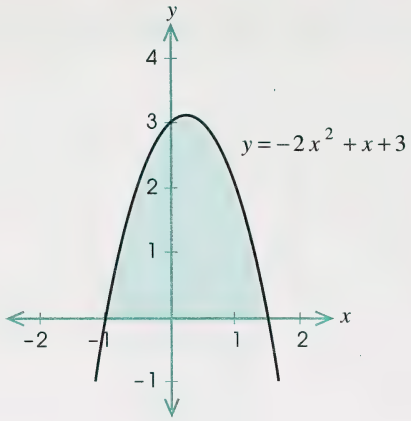
The area is approximately 5.7.

d.



$$
\begin{aligned}
{}_0A_{2\pi} &= \int_0^{2\pi} \sin\frac{x}{2}\,dx \\
&= \left[-2\cos\frac{x}{2}\right]_0^{2\pi} \\
&= (-2\cos\pi) - (-2\cos 0) \\
&= -2(-1) - [-2(1)] \\
&= 2 + 2 \\
&= 4
\end{aligned}
$$

2. a.



Let $-2x^2 + x + 3 = 0$.

$$2x^2 - x - 3 = 0$$

$$(x+1)(2x-3) = 0$$

$x + 1 = 0$ or $2x - 3 = 0$

$x = -1$ $\qquad$ $x = \frac{3}{2}$

$$_{-1}A_{\frac{3}{2}} = \int_{-1}^{\frac{3}{2}} \left(-2x^2 + x + 3\right)\,dx$$

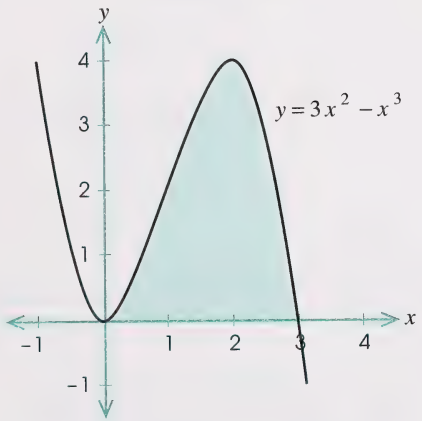
$$= \left[-\frac{2x^3}{3} + \frac{x^2}{2} + 3x\right]_{-1}^{\frac{3}{2}}$$

$$= -\frac{2\left(\frac{3}{2}\right)^3}{3} + \frac{\left(\frac{3}{2}\right)^2}{2} + 3\left(\frac{3}{2}\right) + \frac{2(-1)^3}{3} - \frac{(-1)^2}{2} - 3(-1)$$

$$= \frac{125}{24} \text{ or } 5\frac{5}{24}$$

The area is $5\frac{5}{24}$.

b.



Let $3x^2 - x^3 = 0$.

$$x^2(3 - x) = 0$$

$$x^2 = 0 \text{ or } 3 - x = 0$$

$$x = 0 \qquad x = 3$$

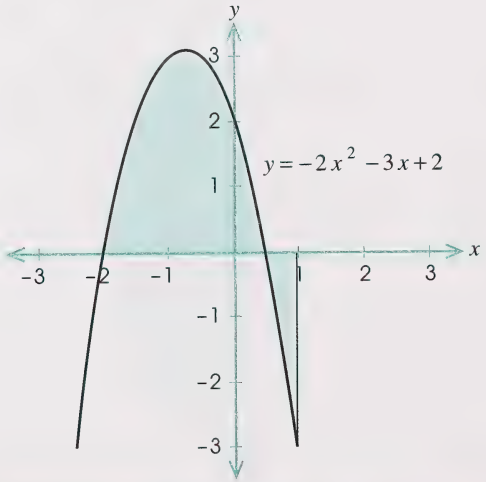
$$\int_0^3 \left(3x^2 - x^3\right)\,dx = \left[x^3 - \frac{x^4}{4}\right]_0^3$$

$$= 27 - \frac{81}{4} - 0$$

$$= \frac{27}{4} \text{ or } 6\frac{3}{4}$$

The area is $6\frac{3}{4}$.

3. a.



Let $-2x^2 - 3x + 2 = 0$.

$$2x^2 + 3x - 2 = 0$$
$$(x+2)(2x-1) = 0$$
$$x = -2 \text{ or } x = \frac{1}{2}$$

$$-2A_1 = \left|\int_{-2}^{\frac{1}{2}} \left(-2x^2 - 3x + 2\right) dx\right| + \left|\int_{\frac{1}{2}}^{1} \left(-2x^2 - 3x + 2\right) dx\right|$$

$$= \left|-\frac{2}{3}x^3 - \frac{3}{2}x^2 + 2x\right|_{-2}^{\frac{1}{2}} + \left|-\frac{2}{3}x^3 - \frac{3}{2}x^2 + 2x\right|_{\frac{1}{2}}^{1}$$

$$= \left|-\frac{2}{3}\left(\frac{1}{8}\right) - \frac{3}{2}\left(\frac{1}{4}\right) + 2\left(\frac{1}{2}\right) - \left(-\frac{2}{3}\right)(-2)^3 + \frac{3}{2}(-2)^2 - 2(-2)\right|$$
$$+ \left|-\frac{2}{3} - \frac{3}{2} + 2 + \frac{2}{3}\left(\frac{1}{2}\right)^3 + \frac{3}{2}\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)\right|$$

$$= \left|\frac{125}{24}\right| + \left|-\frac{17}{24}\right|$$

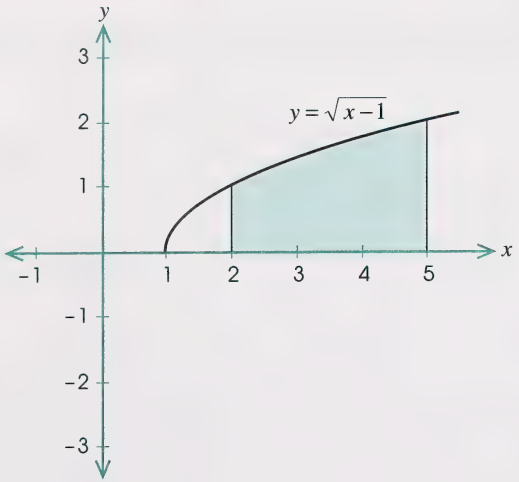
$$= \left|5\frac{5}{24}\right| + \left|\frac{17}{24}\right|$$

$$= 5\frac{22}{24}$$

$$= 5\frac{11}{12}$$

The area is $5\frac{11}{12}$.

b.



$$\int_2^5 \sqrt{x-1}\,dx = \int_2^5 (x-1)^{\frac{1}{2}}\,dx$$

Assume the primitive is of the following form:

$$F(x) = \frac{a(x-1)^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= \frac{2}{3}a(x-1)^{\frac{3}{2}}$$

$$F'(x) = \frac{2}{3}a\left(\frac{3}{2}\right)(x-1)^{\frac{1}{2}}(1)$$

$$= a(x-1)^{\frac{1}{2}}$$

Solve for a.

$$a(x-1)^{\frac{1}{2}} = (x-1)^{\frac{1}{2}}$$

$$a = 1$$

$$\therefore\ F(x) = \frac{2}{3}(x-1)^{\frac{3}{2}}$$

$$\int_2^5 \left(\sqrt{x-1}\right)\,dx = \frac{2}{3}\left[(x-1)^{\frac{3}{2}}\right]_2^5$$

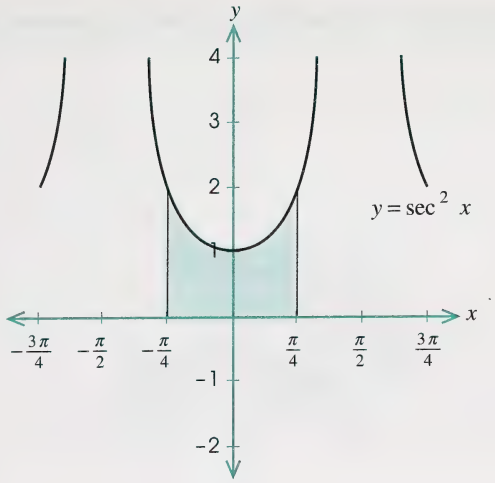
$$= \frac{2}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$

$$= \frac{2}{3}(7)$$

$$= 4\frac{2}{3}$$

The area is $4\frac{2}{3}$.

c.



$$_{-\frac{\pi}{4}}A_{\frac{\pi}{4}} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x \, dx$$

$$= [\tan x]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

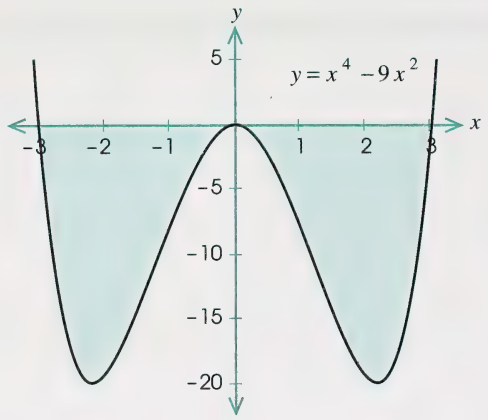
$$= \tan\left(\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{4}\right)$$

$$= 1 - (-1)$$

$$= 2$$

The area is 2.

4. a.



Let $x^4 - 9x^2 = 0$.

$$x^2\left(x^2 - 9\right) = 0$$

$$x^2(x-3)(x+3) = 0$$

$x^2 = 0$ or $x - 3 = 0$ or $x + 3 = 0$

$x = 0$ $\quad$ $x = 3$ $\quad$ $x = -3$

$$\int_{-3}^{3}\left(x^4 - 9x^2\right)\,dx = \left|\frac{x^5}{5} - \frac{9}{3}x^3\right|_{-3}^{3}$$

$$= \left|\frac{3^5}{5} - 3(3)^3 - \frac{(-3)^5}{5} + 3(-3)^3\right|$$

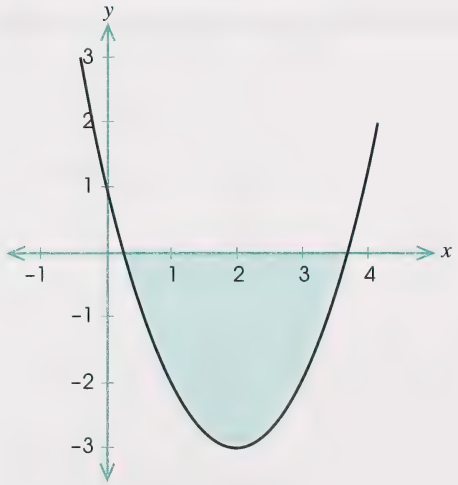
$$= \left|\frac{243}{5} - 81 + \frac{243}{5} - 81\right|$$

$$= \left|-\frac{324}{5}\right|$$

$$= 64\frac{4}{5}$$

The area is $64\frac{4}{5}$.

b.



Let $(x-2)^2 - 3 = 0$.

$$x^2 - 4x + 4 - 3 = 0$$

$$x^2 - 4x + 1 = 0$$

$$x = \frac{4 \pm \sqrt{16-4}}{2}$$

$$= 2 \pm \sqrt{3}$$

$$\int_{2-\sqrt{3}}^{2+\sqrt{3}}\left[(x-2)^2-3\right]dx=\int_{2-\sqrt{3}}^{2+\sqrt{3}}\left(x^2-4x+1\right)dx$$

$$=\left|\frac{x^3}{3}-2x^2+x\right|_{2-\sqrt{3}}^{2+\sqrt{3}}$$

$$=\left|\frac{\left(2+\sqrt{3}\right)^3}{3}-2\left(2+\sqrt{3}\right)^2+2+\sqrt{3}-\frac{\left(2-\sqrt{3}\right)^3}{3}+2\left(2-\sqrt{3}\right)^2-\left(2-\sqrt{3}\right)\right|$$

$$=\left|\frac{8+12\sqrt{3}+18+3\sqrt{3}}{3}-2\left(4+4\sqrt{3}+3\right)+2+\sqrt{3}-\frac{\left(8-12\sqrt{3}+18-3\sqrt{3}\right)}{3}+2\left(4-4\sqrt{3}+3\right)-2+\sqrt{3}\right|$$

$$=\left|\frac{24\sqrt{3}+6\sqrt{3}}{3}-14\sqrt{3}\right|$$

$$=\left|8\sqrt{3}+2\sqrt{3}-14\sqrt{3}\right|$$

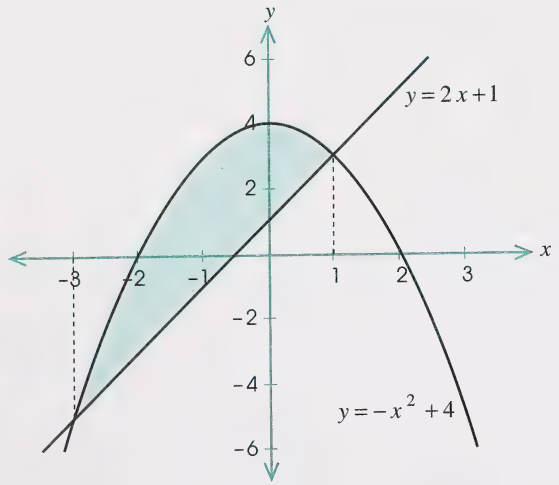
$$=\left|-4\sqrt{3}\right|$$

$$=4\sqrt{3}$$

The area is $4\sqrt{3}$.

Section 2: Activity 5

1. a.



$$\begin{cases} y = -x^2 + 4 \\ y = 2x + 1 \end{cases}$$

$-x^2 + 4 = 2x + 1$ (Find where the graphs intersect.)

$x^2 + 2x - 3 = 0$

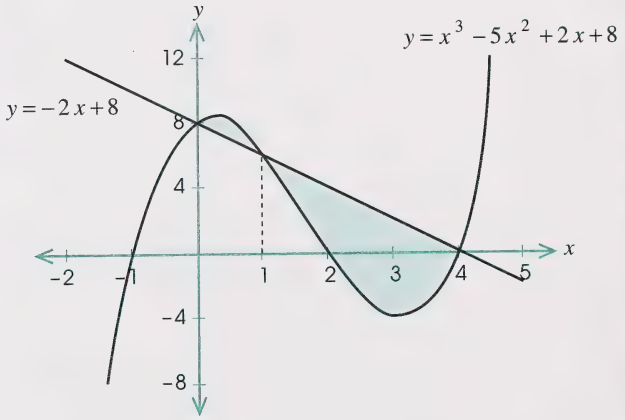
$(x+3)(x-1) = 0$

$x + 3 = 0$ or $x - 1 = 0$

$x = -3$ $\quad$ $x = 1$

$$
\begin{aligned}
{}_{-3}A_1 &= \int_{-3}^{1} \left[\left(-x^2 + 4\right) - (2x + 1) \right] dx \\
&= \int_{-3}^{1} \left(-x^2 - 2x + 3\right) dx \\
&= \left[\frac{-x^3}{3} - x^2 + 3x \right]_{-3}^{1} \\
&= -\frac{1}{3} - 1 + 3 + \frac{(-3)^3}{3} + (-3)^2 - 3(-3) \\
&= -\frac{1}{3} - 1 + 3 - 9 + 9 + 9 \\
&= 10\frac{2}{3}
\end{aligned}
$$

b.



$$\begin{cases} y = x^3 - 5x^2 + 2x + 8 \\ y = -2x + 8 \end{cases}$$

$$x^3 - 5x^2 + 2x + 8 = -2x + 8$$
$$x^3 - 5x^2 + 4x = 0$$
$$x\left(x^2 - 5x + 4\right) = 0$$
$$x(x-1)(x-4) = 0$$
$$x = 0 \text{ or } x - 1 = 0 \text{ or } x - 4 = 0$$
$$x = 1 \qquad x = 4$$

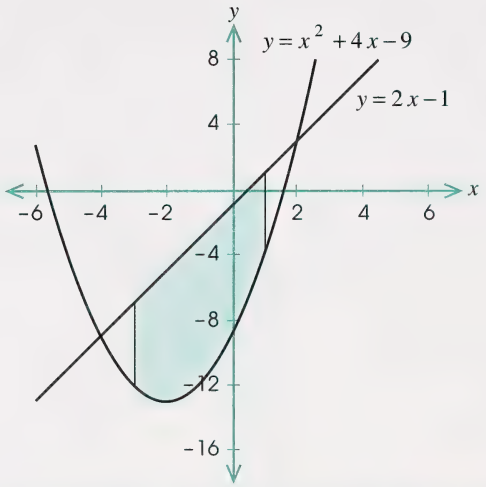
$${}_0A_4 = {}_0A_1 + {}_1A_4$$

$${}_0A_1 = \int_0^1 \left[\left(x^3 - 5x^2 + 2x + 8\right) - (-2x + 8)\right] dx$$
$$= \left[\frac{x^4}{4} - \frac{5x^3}{3} + x^2 + 8x + x^2 - 8x\right]_0^1$$
$$= \frac{1}{4} - \frac{5}{3} + 2$$
$$= \frac{7}{12}$$

$${}_1A_4 = \int_1^4 \left[(-2x + 8) - \left(x^3 - 5x^2 + 2x + 8\right)\right] dx$$
$$= \int_1^4 \left(-x^3 + 5x^2 - 4x\right) dx$$
$$= \left[-\frac{x^4}{4} + \frac{5x^3}{3} - 2x^2\right]_1^4$$
$$= -64 + \frac{320}{3} - 32 + \frac{1}{4} - \frac{5}{3} + 2$$
$$= -96 + 105 + \frac{1}{4} + 2$$
$$= 11\frac{1}{4}$$

The total area is $\frac{7}{12} + 11\frac{1}{4} = 11\frac{5}{6}$.

2.



$$\begin{cases} y = x^2 + 4x - 9 \\ y = 2x - 1 \end{cases}$$

$$x^2 + 4x - 9 = 2x - 1$$

$$x^2 + 2x - 8 = 0$$

$$(x-2)(x+4) = 0$$

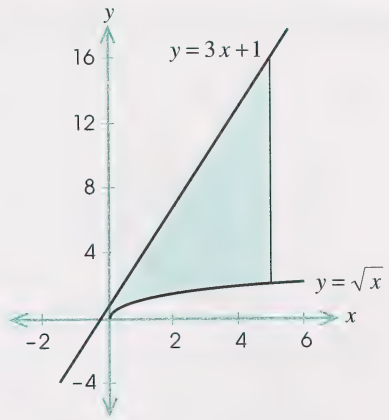
$$x - 2 = 0 \quad \text{or} \quad x + 4 = 0$$

$$x = 2 \qquad x = -4$$

Note that the required area is for the interval $-3 \le x \le 1$. It is **not** the area between the intersection points.

$$\begin{aligned} {}_{-3}A_1 &= \int_{-3}^{1} \left[(2x-1) - \left(x^2 + 4x - 9 \right) \right] dx \\ &= \int_{-3}^{1} \left(-x^2 - 2x + 8 \right) dx \\ &= \left[-\frac{x^3}{3} - x^2 + 8x \right]_{-3}^{1} \\ &= -\frac{1}{3} - 1 + 8 + \frac{(-3)^3}{3} + (-3)^2 - 8(-3) \\ &= -\frac{1}{3} - 1 + 8 - 9 + 9 + 24 \\ &= 30\frac{2}{3} \end{aligned}$$

3.



Let $3x + 1 = \sqrt{x}$.

$$9x^2 + 6x + 1 = x$$

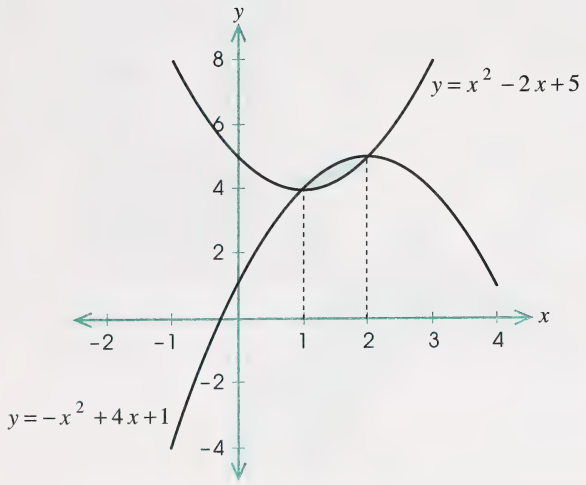
$$9x^2 + 5x + 1 = 0$$

$$x = \frac{-5 \pm \sqrt{25 - 36}}{18}$$

Therefore, there is no point of intersection.

$$
\begin{aligned}
{}_0A_5 &= \int_0^5 \left[(3x+1)-\sqrt{x}\right] dx \\
&= \left[\frac{3x^2}{2}+x-\frac{2x^{\frac{3}{2}}}{3}\right]_0^5 \\
&= \frac{3(5)^2}{2}+(5)-\frac{2(5)^{\frac{3}{2}}}{3}-0 \\
&= \frac{75}{2}+5-\frac{10\sqrt{5}}{3} \\
&= 42\frac{1}{2}-\frac{10}{3}\sqrt{5} \\
&\doteq 35.05
\end{aligned}
$$

4.

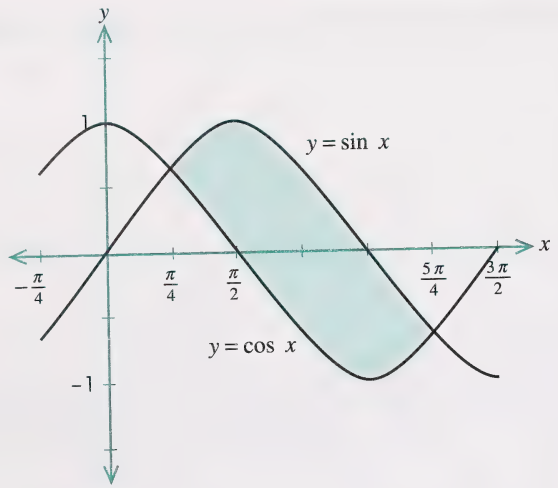


$$
\begin{cases} y = x^2 - 2x + 5 \\ y = -x^2 + 4x + 1 \end{cases}
$$

$$
\begin{aligned}
x^2 - 2x + 5 &= -x^2 + 4x + 1 \\
2x^2 - 6x + 4 &= 0 \\
2\left(x^2 - 3x + 2\right) &= 0 \\
2(x-1)(x-2) &= 0 \\
x-1=0 \quad &\text{or} \quad x-2=0 \\
x=1 \quad &\qquad x=2
\end{aligned}
$$

$$
\begin{aligned}
{}_1A_2 &= \int_1^2 \left[\left(-x^2+4x+1\right)-\left(x^2-2x+5\right)\right] dx \\
&= \int_1^2 \left(-2x^2+6x-4\right) dx \\
&= \left[-\frac{2}{3}x^3+3x^2-4x\right]_1^2 \\
&= -\frac{2}{3}(2)^3+3(2)^2-4(2)+\frac{2}{3}(1)^3-3(1)^2+4(1) \\
&= -\frac{16}{3}+12-8+\frac{2}{3}-3+4 \\
&= 5\frac{2}{3}-5\frac{1}{3} \\
&= \frac{1}{3}
\end{aligned}
$$

5. a.



$$\begin{cases} y = \sin x \\ y = \cos x \end{cases}$$

$$\sin x = \cos x$$

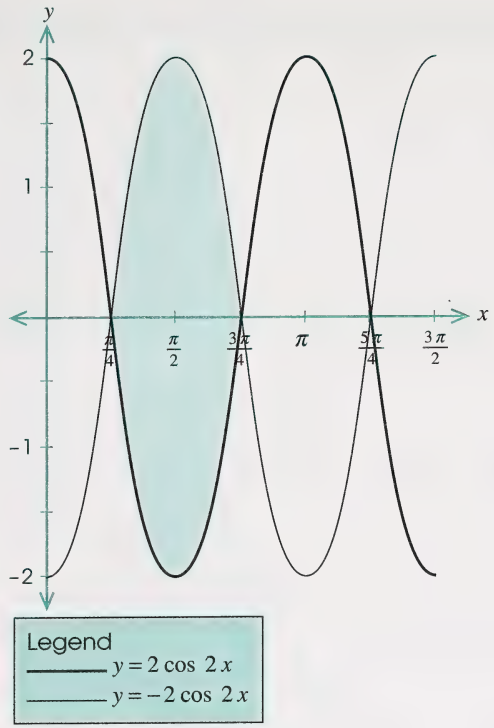
$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$

Between 0 and 2π, $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$. Therefore, the limits of integration occur at the points of intersection.

$$\begin{aligned}
{}_{\frac{\pi}{4}}A_{\frac{5\pi}{4}} &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\, dx \\
&= \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\
&= \left(-\cos \frac{5\pi}{4} - \sin \frac{5\pi}{4}\right) - \left(-\cos \frac{\pi}{4} - \sin \frac{\pi}{4}\right) \\
&= \left[-\left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\right] - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) \\
&= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \left(-\sqrt{2}\right) \\
&= \sqrt{2} + \sqrt{2} \\
&= 2\sqrt{2}
\end{aligned}$$

b.



$$\begin{cases} y = 2\cos 2x \\ y = -2\cos 2x \end{cases}$$

$$2\cos 2x = -2\cos 2x$$
$$4\cos 2x = 0$$
$$\cos 2x = 0$$

Between $x = 0$ and $x = \pi$, $\cos 2x = 0$ at $x = \frac{\pi}{4}$ and $\frac{3\pi}{4}$. These are also the limits of integration.

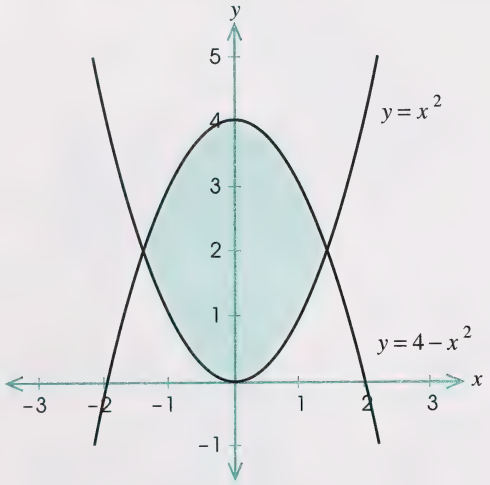
$$\begin{aligned} {}_{\frac{\pi}{4}}A_{\frac{3\pi}{4}} &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (-2\cos 2x - 2\cos 2x)\, dx \\ &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (-4\cos 2x)\, dx \\ &= \left[-2\sin 2x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\ &= -2\sin 2\left(\frac{3\pi}{4}\right) - \left[-2\sin 2\left(\frac{\pi}{4}\right)\right] \\ &= -2\sin\frac{3\pi}{2} + 2\sin\frac{\pi}{2} \\ &= -2(-1) + 2(1) \\ &= 2 + 2 \\ &= 4 \end{aligned}$$

Section 2: Follow-up Activities

Extra Help

Part A

1.

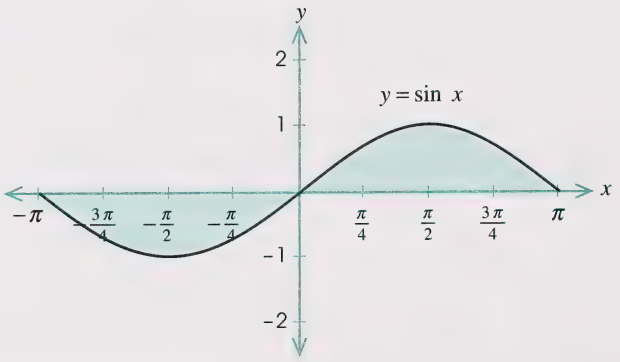


$$\begin{cases} y = 4 - x^2 \\ y = x^2 \end{cases}$$

$$x^2 = 4 - x^2$$
$$2x^2 = 4$$
$$x = \pm\sqrt{2}$$

$$-\sqrt{2}A_{\sqrt{2}} = 2\ {}_0A_{\sqrt{2}}$$
$$= 2\int_0^{\sqrt{2}} \left(4 - x^2 - x^2\right) dx$$
$$= 2\int_0^{\sqrt{2}} \left(4 - 2x^2\right) dx$$
$$= 2\left[4x - \frac{2x^3}{3}\right]_0^{\sqrt{2}}$$
$$= 2\left\{\left[4\sqrt{2} - \frac{2\left(\sqrt{2}\right)^3}{3}\right] - \left[4(0) - \frac{2(0)^3}{3}\right]\right\}$$
$$= 2\left\{4\sqrt{2} - \frac{4\sqrt{2}}{3}\right\}$$
$$= \frac{16\sqrt{2}}{3}$$

2.



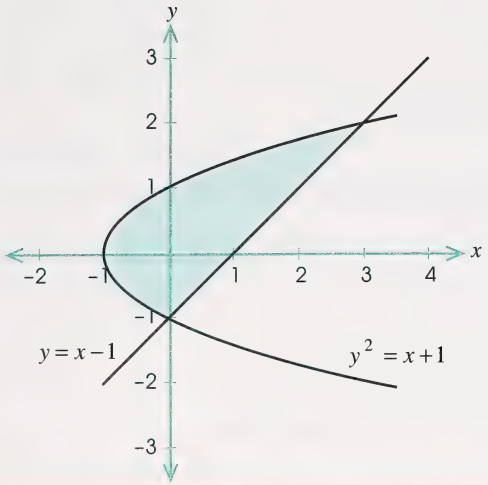
The graph of $y = \sin\ x$ is symmetric about the origin. The shaded regions are equal in area.

$$
\begin{aligned}
{}_{-\pi}A_{\pi} &= 2\int_0^{\pi} \sin\ x\, dx \\
&= \left[-2\cos\ x\right]_0^{\pi} \\
&= -2(\cos\ \pi - \cos\ 0) \\
&= -2(-1-1) \\
&= 4
\end{aligned}
$$

Part B

There were no questions in Part B.

Enrichment



$$
\begin{cases} y^2 = x+1 \\ y = x-1 \end{cases}
$$

$$
\begin{aligned}
y^2 &= y^2 \\
\therefore\ (x-1)^2 &= x+1 \\
x^2 - 2x + 1 &= x+1 \\
x^2 - 3x &= 0 \\
x(x-3) &= 0
\end{aligned}
$$

$x = 0$ or $x = 3$

Thus, $(0,-1)$ and $(3,2)$ are the two points of intersection. The extremes are $(-1,0)$ and $(3,2)$.

$$
\begin{aligned}
&\int_{-1}^{0}\left(\sqrt{x+1}+\sqrt{x+1}\right)dx + \int_0^3\left[\sqrt{x+1}-(x-1)\right]dx \\
&= \int_{-1}^{0}\left(2\sqrt{x+1}\right)dx + \int_0^3\left(\sqrt{x+1}-x+1\right)dx \\
&= \left[\frac{4}{3}(x+1)^{\frac{3}{2}}\right]_{-1}^{0} + \left[\frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{x^2}{2} + x\right]_0^3 \\
&= \frac{4}{3}(0+1)^{\frac{3}{2}} - \frac{4}{3}(0)^{\frac{3}{2}} + \frac{2}{3}(3+1)^{\frac{3}{2}} - \frac{(3)^2}{2} + 3 - \frac{2}{3}(0+1)^{\frac{3}{2}} + 0 - 0 \\
&= \frac{4}{3} + \frac{16}{3} - \frac{9}{2} + 3 - \frac{2}{3} \\
&= 4\frac{1}{2}
\end{aligned}
$$

Section 3: Activity 1

1. a. $v=(1-2t)^3$ and $s=1$ when $t=0$

$$\frac{ds}{dt}=(1-2t)^3$$
$$s=\int(1-2t)^3\ dt+C$$

Assume the antiderivative is of the form $s=\frac{a(1-2t)^4}{4}$; then differentiate.

$$\frac{ds}{dt}=\frac{4a(1-2t)^3}{4}(-2)$$
$$=-2a(1-2t)^3$$

Solve for a.

$$-2a(1-2t)^3=(1-2t)^3$$
$$-2a=1$$
$$a=-\frac{1}{2}$$

$$\therefore\ s=-\frac{(1-2t)^4}{8}+C$$

Substitute $s=1$ and $t=0$ into the equation.

$$1=-\frac{[1-2(0)]^4}{8}+C$$
$$1=-\frac{1}{8}+C$$
$$C=\frac{9}{8}$$

$$\therefore\ s=-\frac{(1-2t)^4}{8}+\frac{9}{8}$$

b. $v=(t-1)^{\frac{1}{2}}$ and $s=4$ when $t=2$

$$\frac{ds}{dt}=(t-1)^{\frac{1}{2}}$$

$$s=\int(t-1)^{\frac{1}{2}}\ dt+C$$
$$=\frac{2(t-1)^{\frac{3}{2}}}{3}+C$$

Substitute $s=4$ and $t=2$ into the equation.

$$4 = \frac{2(2-1)^{\frac{3}{2}}}{3} + C$$

$$4 = \frac{2(1)^{\frac{3}{2}}}{3} + C$$

$$C = 4 - \frac{2}{3}$$

$$= \frac{10}{3}$$

$$\therefore\ s = \frac{2(t-1)^{\frac{3}{2}}}{3} + \frac{10}{3}$$

c. $v = 2\cos 3t$ and $s = -1$ when $t = \frac{\pi}{6}$

$$s = \int 2\cos 3t\, dt + C$$

$$= \frac{2}{3}\sin 3t + C$$

Substitute $s = -1$ when $t = \frac{\pi}{6}$.

$$\therefore\ s = \frac{2}{3}\sin\frac{3\pi}{6} + C$$

$$-1 = \frac{2}{3}\sin\left(\frac{\pi}{2}\right) + C$$

$$-1 = \frac{2}{3}(1) + C$$

$$C = -\frac{5}{3} \qquad \therefore\ s = \frac{2}{3}\sin 3t - \frac{5}{3}$$

2. The object changes direction when $v = 0$.

$$3t^2 - 6t = 0$$

$$3t(t-2) = 0$$

$$3t = 0 \text{ or } t - 2 = 0$$

$$t = 0 \qquad t = 2$$

Total Distance $= {}_0A_2 + {}_2A_3$

$$= \int_0^2 \left(3t^2 - 6t\right) dt + \int_2^3 \left(3t^2 - 6t\right) dt$$

$$= \left|t^3 - 3t^2\right|_0^2 + \left[t^3 - 3t^2\right]_2^3$$

$$= \left|\left[2^3 - 3(2)^2\right] - \left[0^3 - 3(0)^2\right]\right| + \left[3^3 - 3(3)^2\right] - \left[2^3 - 3(2)^2\right]$$

$$= |8 - 12 - 0| + (27 - 27) - (8 - 12)$$

$$= |-4| + 0 - (-4)$$

$$= 8$$

3. Since $v = 2 - 10t$, $s = \int (2 - 10t)\, dt + C$

$$= 2t - 5t^2 + C$$

Substitute $t = 0$ and $s = 100$ into the equation.

$$100 = 2(0) - 5(0)^2 + C$$

$$C = 100$$

$\therefore\ s = 2t - 5t^2 + 100$

The bag will strike the ground when $s = 0$.

$$0 = 2t - 5t^2 + 100$$

$$5t^2 - 2t - 100 = 0$$

Use the quadratic formula.

$$t = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(5)(-100)}}{2(5)}$$

$$= \frac{2 \pm \sqrt{4 + 2000}}{10}$$

Since time is non-negative, $t = \frac{2 + \sqrt{2004}}{10}$

$$= \frac{2 + \sqrt{4(501)}}{10}$$

$$= \frac{2 + 2\sqrt{501}}{10}$$

$$= \frac{1 + \sqrt{501}}{5}$$

The sandbag will strike the ground in about 4.68 s.

4. Since $v = 40 - 5t^{\frac{3}{2}}$, $s = \int \left(40 - 5t^{\frac{3}{2}} \right) dt + C$

$$= 40t - 5\left(\frac{2}{5}\right)t^{\frac{5}{2}} + C$$

$$= 40t - 2t^{\frac{5}{2}} + C$$

When $t = 0$, $s = 0$, and $C = 0$.

$\therefore\ s = 40t - 2t^{\frac{5}{2}}$

The car will stop when $v = 0$.

$$40 - 5t^{\frac{3}{2}} = 0$$

$$5t^{\frac{3}{2}} = 40$$

$$t^{\frac{3}{2}} = 8$$

$$t = 8^{\frac{2}{3}}$$

$$= 4 \text{ s}$$

$$\therefore\ s = 40(4) - 2(4)^{\frac{5}{2}}$$

$$= 160 - 64$$

$$\doteq 96$$

The stopping distance is 96 m.

Section 3: Activity 2

1. Determine the velocity function.

Assume $a = -10 \text{ m/s}^2$.

$$\begin{aligned} v &= \int a\, dt + C_1 \\ &= \int (-10)\, dt + C_1 \\ &= -10t + C_1 \end{aligned}$$

When $t = 0$ s, $v = -100$ m/s.

$$\begin{aligned} -100 &= -10(0) + C_1 \\ C_1 &= -100 \end{aligned}$$

$\therefore v = -10t - 100$

The velocity is negative because the plane is flying straight downward.

Next, find the position function.

$$\begin{aligned} h &= \int (-10t - 100)\, dt + C_2 \\ &= -5t^2 - 100t + C_2 \end{aligned}$$

When $t = 0$ s, $h = 500$ m.

$$\begin{aligned} 500 &= -5(0)^2 - 100(0) + C_2 \\ C_2 &= 500 \end{aligned}$$

$\therefore h = -5t^2 - 100t + 500$

The bolt strikes the ground when $h = 0$ m.

$$\begin{aligned} 0 &= -5t^2 - 100t + 500 \\ 5t^2 + 100t - 500 &= 0 \\ 5\left(t^2 + 20t - 100\right) &= 0 \end{aligned}$$

Use the quadratic formula.

$$\begin{aligned} t &= \frac{-20 \pm \sqrt{(20)^2 - 4(1)(-100)}}{2(1)} \\ &= \frac{-20 \pm \sqrt{800}}{2} \\ &= \frac{-20 \pm 20\sqrt{2}}{2} \\ &= -10 \pm 10\sqrt{2} \end{aligned}$$

Because time is non-negative, $t = -10 + 10\sqrt{2}$ s. Substitute $t = -10 + 10\sqrt{2}$ into $v = -10t - 100$ to find the velocity when the bolt hits the ground.

$$\begin{aligned} v &= -10\left(-10 + 10\sqrt{2}\right) - 100 \\ &= -100\sqrt{2} \\ &\doteq 141.42 \end{aligned}$$

The bolt hits the ground at a velocity of about 141.42 m/s.

2. Assume $a = -10 \text{ m/s}^2$.

$$\begin{aligned} v &= \int a\,dt + C_1 \\ &= \int (-10)\,dt + C_1 \\ &= -10t + C_1 \end{aligned}$$

When $t = 0$ s, $v = 20$ m/s.

The velocity is positive because the ball is tossed upward.

$$\begin{aligned} 20 &= -10(0) + C_1 \\ C_1 &= 20 \end{aligned}$$

$\therefore\ v = -10t + 20$

The ball reaches its maximum height when $v = 0$.

$$\begin{aligned} -10t + 20 &= 0 \\ t &= 2 \end{aligned}$$

Next, determine the position function.

$$\begin{aligned} h &= \int (-10t + 20)\,dt + C_2 \\ &= -5t^2 + 20t + C_2 \end{aligned}$$

When $t = 0$ s, $h = 2$ m.

$$\begin{aligned} 2 &= -5(0)^2 + 20(0) + C_2 \\ C_2 &= 2 \end{aligned}$$

$\therefore\ h = -5t^2 + 20t + 2$

To find the maximum height, find h when $t = 2$ s.

$$\begin{aligned} h &= -5(2)^2 + 20(2) + 2 \\ &= 22 \end{aligned}$$

The maximum height is 22 m.

3. Since the car decelerates at 3 m/s^2, use $a = -3$.

$$\begin{aligned} v &= \int a\,dt + C_1 \\ &= \int (-3)\,dt + C_1 \\ &= -3t + C_1 \end{aligned}$$

When $t = 0$ s, $v = 30$ m/s.

$$\begin{aligned} 30 &= -3(0) + C_1 \\ C_1 &= 30 \end{aligned}$$

$\therefore\ v = -3t + 30$

When the car stops, $v = 0$ m/s.

$$0 = -3t + 30$$
$$t = 10$$

The car takes 10 s to stop.

To determine the stopping distance, find the position function d.

$$d = \int v\, dt + C_2$$
$$= \int (-3t + 30)\, dt$$
$$= -\frac{3}{2}t^2 + 30t + C_2$$

When $t = 0$ s, $d = 0$ m.

$$0 = -\frac{3}{2}(0)^2 + 30(0) + C_2$$
$$C_2 = 0$$

$$\therefore\ d = -\frac{3}{2}t^2 + 30t$$

Since the initial position is 0 m and the car does not change direction, the stopping distance is simply the final position.

When $t = 10$ s, $d = -\frac{3}{2}(10)^2 + 30(10)$

$$= 150$$

The car takes 150 m to stop.

4. Recall $v = at + v_0$.

$$d = \frac{1}{2}at^2 + v_0 t + d_0$$

In Example 2, $a = -10$ m/s^2, $v_0 = 2$ m/s, and $d_0 = 1.5$ m.

Therefore, $v = -10t + 2$ and $d = -5t^2 + 2t + 1.5$.

5. In Example 3, you derived $v = at + v_0$.

$$\therefore\ v^2 = (at + v_0)^2$$
$$v^2 = a^2t^2 + 2atv_0 + {v_0}^2$$
$$v^2 = {v_0}^2 + a^2t^2 + 2atv_0$$
$$v^2 = {v_0}^2 + 2a\left(\frac{1}{2}at^2 + v_0 t\right)$$

Now, $d = \frac{1}{2}at^2 + v_0 t + d_0$

$$d - d_0 = \frac{1}{2}at^2 + v_0 t$$

$\therefore\ v^2 = {v_0}^2 + 2a(d - d_0)$

$\therefore\ v^2 = {v_0}^2 + 2as$

Remember: s = displacement

6. $v = \int a\,dt + C_1$

$= \int (-2t)\,dt + C_1$

$= -t^2 + C_1$

At $t = 0$, $v = 2$

$2 = -0^2 + C_1$

$C_1 = 2$

$\therefore\ v = -t^2 + 2$

$s = \int \left(-t^2 + 2\right) dt + C_2$

$= -\frac{1}{3}t^3 + 2t + C_2$

At $t = 0$, $s = 8$

$8 = -\frac{1}{3}(0)^3 + 2(0) + C_2$

$C_2 = 8$

$\therefore\ s = -\frac{1}{3}t^3 + 2t + 8$

7. $v = \int (3t + 2)\,dt + C_1$

$= \frac{3}{2}t^2 + 2t + C_1$

At $t = 0$ s, $v = 0$ m/s.

$0 = \frac{3}{2}(0)^2 + 2(0) + C_1$

$C_1 = 0$

$\therefore\ v = \frac{3}{2}t^2 + 2t$

$h = \int \left(\frac{3}{2}t^2 + 2t\right) dt + C_2$

$= \frac{t^3}{2} + t^2 + C_2$

Since you do not know the initial position, call it h_0.

$h = \frac{t^3}{2} + t^2 + h_0$

At $t = 0$ s, $h = h_0$.

At $t = 3$ s, $h = \frac{3^3}{2} + 3^2 + h_0$

$= 21.5 + h_0$

Therefore, the rocket travelled 21.5 m in 3 s.

Section 3: Activity 3

1. a. The maximum and the minimum displacement occur when $\frac{dx}{dt} = 0$.

Since $x = -4 \cos \pi t$, $\frac{dx}{dt} = 4\pi \sin \pi t$.

Now, $4\pi \sin \pi t = 0$

$\sin \pi t = 0$

Now $\sin \phi = 0$ when $\phi = n\pi$, where $n = 0, 1, 2, \ldots$

$\therefore \pi t = n\pi$, where $n = 0, 1, 2, \ldots$

$t = n$, where $n = 0, 1, 2, \ldots$

When $n = 0$, $t = 0$ s and $x = -4 \cos 0$

$= -4(1)$

$= -4$

When $n = 1$, $t = 1$ s and $x = -4 \cos \pi$

$= -4(-1)$

$= 4$

When $n = 2$, $t = 2$ s and $x = -4 \cos 2\pi$

$= -4(1)$

$= -4$

When $n = 3$, $t = 3$ s and $x = -4 \cos 3\pi$

$= -4(-1)$

$= 4$

The minimum value of x is -4 cm. This occurs at the following:

$t = 0$ s, 2 s, 4 s, 6 s,...

$= 2n$ s where $n = 0, 1, 2, \ldots$

The maximum value of x is 4 cm. This occurs at the following:

$t = 1$ s, 3 s, 5 s, 7 s,...

$= (2n - 1)$ s, where $n = 0, 1, 2, \ldots$

Note: The minimum and maximum values could have been obtained from the amplitude of the displacement function $x = -4 \cos \pi t$, which is $|-4|$ or 4 cm.

b. The period of the motion is the difference in times between two consecutive maxima or between two consecutive minima. Since maximum values occur every 2 s, the period of the spring is also 2 s.

You could have found the period of the function from what you studied in Mathematics 30. For $\cos b\phi$ the period is $\frac{2\pi}{b}$. The period for $x = -4 \cos \pi t$ is $\frac{2\pi}{\pi}$ or 2 s.

c. The magnitude of the velocity is a maximum when acceleration is zero. Since $\frac{dx}{dt} = 4\pi \sin \pi t$, then

$\frac{d^2x}{dt^2} = 4\pi^2 \cos \pi t$.

The acceleration is zero when $\cos \pi t = 0$. These will be the same times when the displacement is zero.

Now $\cos b\phi = 0$ when $\phi = \frac{\pi}{2} + n\pi$, where $n = 0,\ 1,\ 2,\ldots$

$\therefore\ \pi t = \frac{\pi}{2} + n\pi$, where $n = 0,\ 1,\ 2,\ldots$

$t = \left(\frac{1}{2} + n\right)$ seconds

d. Since $x = -4 \cos \pi t$, then $\frac{d^2x}{dt^2} = 4\pi^2 \cos \pi t$

$= -\pi^2 (-4 \cos \pi t)$

$= -\pi^2 x$

$\therefore\ \frac{d^2x}{dt^2} + \pi^2 x = 0$

2. a. The maximum and minimum displacements must be 0.04 m and -0.04 m, obtained from the amplitude of the displacement function, $s = 0.04 \cos \left(\frac{\pi t}{3}\right)$.

The times when they occur are obtained from $\frac{ds}{dt} = 0$.

$\frac{ds}{dt} = \frac{-0.04\pi}{3} \sin \left(\frac{\pi t}{3}\right)$

Recall that $\sin \phi = 0$ when $\phi = n\pi$, where $n = 0,\ 1,\ 2,\ldots$

$\therefore\ \frac{\pi t}{3} = \pi t$, where $n = 0,\ 1,\ 2,\ldots$

$t = 3t$

When $n = 0$, $t = 0$ and $s = 0.04 \cos 0$

$= 0.04$ m (maximum)

When $n = 1$, $t = 3$ and $s = 0.04 \cos \pi$

$= -0.04$ m (minimum)

The maxima and minima alternate every 3 s.

Maxima occur when $t = 0$ s, 6 s, 12 s,...

Minima occur when $t = 3$ s, 9 s, 15 s,...

b. The period of the motion is 6 s.

The period of $\cos b\phi$ is $\frac{2\pi}{b}$.

The period of $s = 0.04 \cos \left(\frac{\pi t}{3}\right)$ is $\frac{2\pi}{\frac{\pi}{3}}$ or 6 s.

c. The magnitude of the velocity is a maximum when acceleration is zero. Since $\frac{ds}{dt} = \frac{-0.04\pi}{3} \sin\left(\frac{\pi t}{3}\right)$, then $\frac{d^2s}{dt^2} = \frac{-0.04\pi^2}{9} \cos\left(\frac{\pi t}{3}\right)$.

The maximum speed occurs when $\cos\left(\frac{\pi t}{3}\right) = 0$. Again, these are the same times as when the displacement is zero.

Now, $\cos\phi = 0$ when $\phi = \frac{\pi}{2} + n\pi$, where $n = 0,\ 1,\ 2,\ldots$

$$\therefore\ \frac{\pi t}{3} = \frac{\pi}{2} + n\pi, \text{ where } n = 0,\ 1,\ 2,\ldots$$
$$t = (1.5 + 3n)\ \text{s}$$

Of course, maximum speeds occur halfway between times for extreme displacements.

d. Since $s = 0.04\cos\left(\frac{\pi t}{3}\right)$, then $\frac{d^2s}{dt^2} = \frac{-0.04\pi^2}{9}\cos\left(\frac{\pi t}{3}\right)$

$$= -\frac{\pi^2}{9}\left[0.04\cos\left(\frac{\pi t}{3}\right)\right]$$
$$= -\frac{\pi^2}{9}s$$

$$\therefore\ \frac{d^2s}{dt^2} + \frac{\pi^2 s}{9} = 0$$

3. a. $x = \sin(3t - 2)$

$$\frac{dx}{dt} = 3\cos(3t - 2)$$

$$\frac{d^2x}{dt^2} = -9\sin(3t - 2)$$
$$\frac{d^2x}{dt^2} = -9x$$
$$\frac{d^2x}{dt^2} + 9x = 0$$

b. $x = 2\sin 3t + 5\cos 3t$

$$\frac{dx}{dt} = 6\cos 3t - 15\sin 3t$$

$$\frac{d^2x}{dt^2} = -18\sin 3t - 45\cos 3t$$
$$\frac{d^2x}{dt^2} = -9(2\sin 3t + 5\cos 3t)$$
$$\frac{d^2x}{dt^2} = -9x$$
$$\frac{d^2x}{dt^2} + 9x = 0$$

4. Since $v = \pi \cos 2\pi t$, then $y = \int v\, dt + C$

$$= \int (\pi \cos 2\pi t)\, dt + C$$

$$= \frac{\pi}{2\pi} \sin 2\pi t + C$$

When $t = 0$, $y = 0$.

$$0 = \frac{1}{2} \sin 0 + C$$

$$C = 0$$

$$\therefore y = \frac{1}{2} \sin 2\pi t$$

The maximum height is the amplitude of the displacement function. Therefore, the maximum height is $\frac{1}{2}$ m.

Next, find the acceleration at any time t.

$$\frac{d^2y}{dt^2} = -2\pi^2 \sin 2\pi t$$

5. Since $\frac{d^2y}{dt^2} = \sin t$, $\frac{dy}{dt} = \int \sin t\, dt + C$

$$= -\cos t + C$$

Since $\frac{dy}{dt} = 3$ at $t = 0$, then $3 = -\cos 0 + C$

$$3 = -1 + C$$

$$C = 4$$

$$\therefore \frac{dy}{dt} = -\cos t + 4$$

$$y = \int (-\cos t + 4)\, dt + K$$

$$= -\sin t + 4t + K$$

Since $y = 0$ at $t = 0$

$$0 = -\sin 0 + 4(0) + K$$

$$K = 0$$

$$\therefore y = -\sin t + 4t$$

Section 3: Activity 4

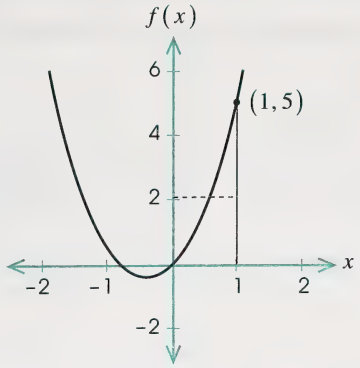
1. $(y_{\text{ave}})_x = \frac{1}{1-0} \int_0^1 (3x^2 + 2x)\, dx$

$$= \int_0^1 (3x^2 + 2x)\, dx$$

$$= \left[x^3 + x^2\right]_0^1$$

$$= (1^3 + 1^2) - (0^3 + 0^2)$$

$$= 2$$



The mean y-value is 2. **Note:** $0 \le (y_{\text{ave}})_x \le 5$

2. $(v_{\text{ave}})_d = \dfrac{1}{\frac{1}{2}gt_2{}^2 - 0}\int_0^{\frac{1}{2}gt_2{}^2} \sqrt{2gd}\, dd$

$$= \frac{2}{gt_2{}^2}\left[\sqrt{2g}\,\frac{d^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{1}{2}gt_2{}^2}$$

$$= \frac{2}{gt_2{}^2}\bullet\frac{2}{3}\bullet\sqrt{2g}\left[\left(\frac{1}{2}gt_2{}^2\right)^{\frac{3}{2}} - 0\right]$$

$$= \frac{4\sqrt{2}\bullet\sqrt{g}g^{\frac{3}{2}}t_2{}^3}{3gt_2{}^2\bullet 2^{\frac{3}{2}}}$$

$$= \frac{4\sqrt{2}\sqrt{g}\bullet g\sqrt{g}\,t_2}{3g\bullet 2\sqrt{2}}$$

$$= \frac{2}{3}gt_2$$

Notice that the average velocity with respect to s is not the same as the average velocity with respect to t, even though the time intervals in both cases are the same.

3. The function you must average over the interval $[0, 10]$ is $f(x) = \sqrt{x}$.

$$\therefore (y_{\text{ave}})_x = \frac{1}{10}\int_0^{10}\sqrt{x}\, dx$$

$$= \frac{1}{10}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^{10}$$

$$= \frac{2}{30}\left[10^{\frac{3}{2}} - 0^{\frac{3}{2}}\right]$$

$$= \frac{2}{30}\left(10^{\frac{3}{2}}\right)$$

$$= \frac{2}{30}\left(10\sqrt{10}\right)$$

$$= \frac{2\sqrt{10}}{3}$$

4. First, find the function.

$T = 20$ at $t = 0$ and $T = 40$ at $t = 50$.

Because the change is uniform, the graph is linear.

$$\text{slope of the line} = \frac{40-20}{50-0}$$
$$= \frac{20}{50}$$
$$= \frac{2}{5}$$

The equation of the line is of the form $T = mt + b$, where $m =$ slope and $b = T$-intercept.

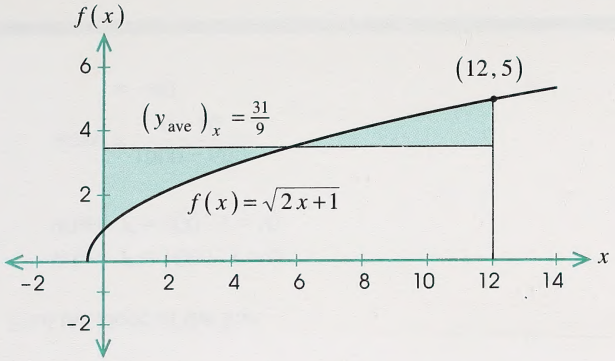
$\therefore\ T = \frac{2}{5}t + 20$, since $m = \frac{2}{5}$ and $b = 20$.

$$\therefore\ \left(T_{\text{ave}}\right)_t = \frac{1}{50-0}\int_0^{50}\left(\frac{2}{5}t + 20\right)dt$$
$$= \frac{1}{50}\left[\frac{1}{5}t^2 + 20t\right]_0^{50}$$
$$= \frac{1}{50}\left[\frac{1}{5}(50)^2 + 20(50)\right]$$
$$= \frac{1}{50}\bullet(50)\left[\frac{1}{5}(50) + 20\right]$$
$$= 10 + 20$$
$$= 30°\text{C}$$

The average temperature with respect to time is 30°C. This value is also the arithmetic mean of 20°C and 40°C. This is the case because the function $T = \frac{2}{5}t + 20$ is linear.

5. a. $\left(y_{\text{ave}}\right)_x = \frac{1}{12-0}\int_0^{12}\sqrt{2x+1}\,dx$

$$= \frac{1}{12}\bullet\left[\frac{(2x+1)^{\frac{3}{2}}}{2\left(\frac{3}{2}\right)}\right]_0^{12}$$
$$= \frac{1}{36}\left[(2x+1)^{\frac{3}{2}}\right]_0^{12}$$
$$= \frac{1}{36}\left[(24+1)^{\frac{3}{2}} - (0+1)^{\frac{3}{2}}\right]$$
$$= \frac{1}{36}\left[25^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$$
$$= \frac{1}{36}[125-1]$$
$$= \frac{124}{36}$$
$$= \frac{31}{9}$$
$$= 3\frac{4}{9}$$



b. $(y_{ave})_x = \frac{1}{2\pi - 0}\int_0^{2\pi} \sin x\, dx$

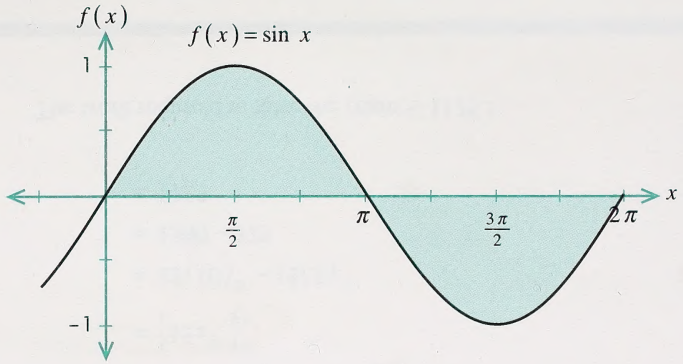
$= \frac{1}{2\pi}[-\cos x]_0^{2\pi}$

$= \frac{1}{2\pi}[-\cos 2\pi - (-\cos 0)]$

$= \frac{1}{2\pi}[-1-(-1)]$

$= 0$

When a function crosses the x-axis, there is some cancellation between positive and negative areas. Here the positive and negative areas are equal in magnitude.



$\therefore (y_{ave})_x = 0$

Section 3: Follow-up Activities

Extra Help

1. displacement = final position – initial position

$= (-2) - (+1)$

$= -3$

The displacement is -3 units.

2. distance = (distance from A to B) – (distance from B to C)

$= 4 + 7$

$= 11$

The distance is 11 units.

3. $\text{average velocity} = \dfrac{\text{displacement}}{\text{time}}$

$$= \frac{-3}{2}$$
$$= -1.5$$

The average velocity is -1.5 units/s.

4. $\text{average speed} = \dfrac{\text{distance}}{\text{time}}$

$$= \frac{11}{2}$$
$$= 5.5$$

The average speed is 5.5 units/s.

Enrichment

1. Find the equation of the force function $F = f(s)$. Since the sand runs out of the bag at a constant rate and the bag is raised at a constant speed, the relationship between F and s is linear.

Find the slope of the line.

When $F = 1000$, $s = 0$.
When $F = 600$, $s = 10$.

$$\text{slope} = \frac{1000 - 600}{0 - 10}$$
$$= -40$$

Use the slope-intercept formula.

$$F = -40s + 1000$$

$$\therefore \text{Work} = \int_0^{10} (-40s + 1000)\, ds$$
$$= \left(-20s^2 + 1000s\right)_0^{10}$$
$$= \left[-20(10)^2 + 1000(10)\right] - \left[-20(0)^2 + 1000(0)\right]$$
$$= -2000 + 10\,000$$
$$= 8000$$

The work required to raise the sandbag is 8000 J.

2. Determine the force function $F = f(s)$. $F = 30s$, where s is in metres.

$$\therefore \text{Work} = \int_5^{10} 30s\, ds$$
$$= \left[15s^2\right]_5^{10}$$
$$= 15(10)^2 - 15(5)^2$$
$$= 1500 - 375$$
$$= 1125$$

The work required to raise the chain is 1125 J.

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